Linear tearing modes in an electron-positron plasma

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Abstract
The general dispersion of tearing modes due to the effects of electron inertia and resistivity in pair plasmas is derived analytically, and is discussed in two cases: $\Delta' \gg 1$ and $\Delta' \ll 1$, where $\Delta'$ is the instability criterion of the tearing mode. It is found that the conditions under which either resistivity or electron inertia dominates depend strongly on the limit of $\Delta'$ considered.

Keywords: electron inertia, resistivity, electron-positron plasmas, tearing modes

(Some figures may appear in colour only in the online journal)

1. Introduction

The tearing mode plays an important role in magnetically confined plasmas of astrophysical and thermonuclear interest. It can change the topology of the magnetic field, and leads to the formation of magnetic islands. At the same time, the energy of the magnetic field is relaxed. Tearing modes can cause macroscopic relaxation events such as solar flares [1] and disruption in magnetically confined laboratory plasmas [2]. Since the well-known Sweet–Parker model [3, 4] and the tearing mode theory [5] were established in the 1960s, a lot of work has been devoted to reconnection. However, collisionless magnetic reconnection always happens in space and high temperature plasmas. Consequently, other terms, with the exception of resistivity, such as electron inertia and electron pressure tensor, in generalized Ohm’s law are important, since they can cause magnetic reconnection. In the four-field and Hamiltonian two-fluid theory [6–9], the non-ideal effects with the exception of resistivity in generalized Ohm’s law, such as electron inertia and temperature, were incorporated into the magnetohydrodynamics. There has been some work focused on the effect of electron inertia in ion-electron plasmas, where a separation of spatial scales between electron and ion flows exists. Otherwise, there is no a separation of spatial scales in pair plasmas, so it is more convenient to investigate the effect of electron inertia at magnetic reconnection. Recently, some simulations and theoretical work have been devoted to reconnection in pair plasmas, which showed that fast magnetic reconnection can happen without fast dispersive waves [10–16].

In pair plasmas, equal masses of the negatively charged electron and positively charged positron eliminate the Hall current in generalized Ohm’s law, and the electron and positron skin depth are identical. Thus, it is of interest to study the physics of collisionless reconnection in pair plasmas. Recently, some attention has been devoted to reconnection in pair plasmas [10–17]. In 1991, the effect of inertia at collisionless tearing mode in ion-electron plasmas has been investigated by Porcelli [18]. It was shown that magnetic reconnection can occur due to inertia. In this article, we will investigate tearing modes for the large-guide-field, low-$\beta$ fluid approximation in pair plasmas, where Hall current disappears in generalized Ohm’s law and only a spatial scale exists, and discuss the conditions under which either electron inertia or resistivity dominates. When tearing modes grow further, the nonlinear effects become important [19]. However, this article focuses on the linear tearing modes in pair plasmas. The technique of reference [20] is used to analyze tearing modes systematically. Here, other effects, such as axial flow and transversal rotation on tearing modes in pair plasmas are not considered.

The paper is organized as follows. In section 2, a set of equations for investigating tearing modes in pair plasmas is presented. The general dispersion of tearing modes is derived and analyzed in section 3. Finally, the conclusion is given in section 4.
2. Basic equations

The momentum equation for each species in a pair plasma is given by

\[
\frac{\partial}{\partial t}(m n \mathbf{v}_p) + \nabla \cdot (m n \mathbf{v}_p \mathbf{v}_p) = -\nabla P_p + n \left( \mathbf{E} + \frac{1}{c} \mathbf{v}_p \times \mathbf{B} \right) + \mathbf{R}_{pe},
\]

(1)

\[
\frac{\partial}{\partial t}(m n \mathbf{v}_e) + \nabla \cdot (m n \mathbf{v}_e \mathbf{v}_e) = -\nabla P_e - n \left( \mathbf{E} + \frac{1}{c} \mathbf{v}_e \times \mathbf{B} \right) + \mathbf{R}_{pe},
\]

(2)

where \( \mathbf{v}_p, P_p \) are the positron velocity and pressure, respectively, \( \mathbf{v}_e, P_e \) are the corresponding electron variables. \( m \) is the mass, and \( \mathbf{R}_{pe} = -\mathbf{R}_{ep} \) is the interspecies friction term. \( \mathbf{E} \) and \( \mathbf{B} \) are the electric and magnetic field, respectively. Here, the quasineutrality \( n_p = n_e = n \) is assumed, and the off-diagonal components of the pressure tensors and viscosity are not considered [17]. Actually, the effect of viscosity was discussed explicitly in reference [16], where it was shown that the viscosity can sustain fast reconnection. Following the procedure of reference [16], we also assume that both species have equal temperature, since both species will reach a self-equilibrium together, and a global equilibrium will result at the same time. Consequently, \( P_p = P_e = P \). By introducing the plasma bulk velocity \( \mathbf{v} = (\mathbf{v}_p + \mathbf{v}_e)/2 \), current density \( \mathbf{J} = ne(\mathbf{v}_p - \mathbf{v}_e) \) and using a resistive model for the friction term \( \mathbf{R}_{pe} = n e \eta \mathbf{J} \), where \( \eta = m n/(n e^2) \) is the plasma resistivity, equations (1) and (2) can be transformed to [16]

\[
\frac{\partial \mathbf{v}}{\partial t} + \mathbf{v} \cdot \nabla \mathbf{v} + \frac{d_e^2}{4} \mathbf{J} \cdot \nabla \mathbf{J} = -\nabla P + \frac{\mathbf{J} \times \mathbf{B}}{2},
\]

(3)

\[
\frac{d_e^2}{2} \frac{\partial \mathbf{I}}{\partial t} + \nabla \cdot (\mathbf{vJ} + \mathbf{Jv}) = \mathbf{E} + \mathbf{v} \times \mathbf{B} - \eta \mathbf{J},
\]

(4)

where the homogeneous density is assumed and the variables have been normalized as follows: \( \mathbf{v} \to \mathbf{v}_\Lambda \mathbf{v}, \quad t \to \tau_\Lambda t, \quad \mathbf{x} \to L \mathbf{x}, \quad \mathbf{B} \to B_\Lambda \mathbf{B}, \quad \eta \to (\tau_\Lambda/\tau_k) \eta \) and \( d_e = c/\omega_B L \), where \( \tau_\Lambda = L^2/(4\pi n_\Lambda), \quad \tau_k = 4\pi n L^2/(\sigma n e^2) \) and \( \omega_B = (4\pi n e^2/m)^{1/2} \) are the characteristic time of Alfvén modes, resistivity diffusion time and plasma frequency, respectively. Here \( d_e \), the electron skin depth, \( L \) is the scale length of equilibrium parameters, \( \tau_\Lambda \) is the Alfvén velocity. Now, we consider tearing modes in a 2D slab configuration in incompressible pair plasmas, where the plasma parameters are independent of the coordinate \( z \) along the dominant equilibrium magnetic field. Thus, in the low-\( \beta \), large-guide-field limit, the magnetic field and velocity field can be written as [16, 17, 21]

\[
\mathbf{B} = \nabla \psi \times \mathbf{e}_z + B_z \mathbf{e}_z, \quad (5)
\]

\[
\mathbf{v} = \nabla \phi \times \mathbf{e}_z, \quad (6)
\]

where \( B_z = \text{const} \), \( \psi, \phi \) are the poloidal flux and stream function, respectively. Here, it is assumed that plasma flow is incompressible in the low-\( \beta \) limit, which implies density \( n \) is constant at all times if it is initially homogeneous. Based on the expressions of magnetic and velocity field, equations (3) and (4) can be derived [17]

\[
\frac{d\psi^2 \phi}{dt} = \frac{1}{2} (\nabla \psi \times \mathbf{e}_z) \cdot \nabla \psi^2 \phi, \quad (7)
\]

\[
\frac{\partial}{\partial t} \left( \eta \psi \frac{d^2 \psi}{\partial x^2} \right) + (\nabla \psi \times \mathbf{e}_z) \cdot \nabla \psi^2 \phi, \quad (8)
\]

Now, the forms of magnetic flux function and stream function can be assumed as

\[
\psi(x, y, t) = \psi_0(x) + \psi_1(x) \cos k y e^{-\gamma t}, \quad \phi(x, y, t) = \phi_1(x) \sin k y e^{-\gamma t},
\]

where \( \gamma \) is the growth rate of tearing modes, and the equilibrium flow is not considered. Thus, equations (7) and (8) can be linearized as

\[
\gamma \left( \frac{\partial^2 \psi_1}{\partial x^2} - k^2 \psi_1 \right) = \frac{1}{2} k \frac{d\psi_0}{dx} \left( \frac{\partial^2 \psi_1}{\partial x^2} - k^2 \psi_1 \right) - k \frac{d\psi_1}{dx} \psi_1, \quad (9)
\]

\[
\gamma \left[ \psi_1 - \frac{d^2 \psi_0}{dx^2} \right] + k \frac{d\psi_0}{dx} \left( \psi_0 - \frac{d^2 \psi_0}{dx^2} \right) \phi_1 = \eta \left( \frac{\partial^2 \psi_1}{\partial x^2} - k^2 \psi_1 \right). \quad (10)
\]

Equations (9) and (10) compose a set of basic equations for investigating tearing modes in incompressible pair plasmas. Next, the boundary-layer theory [22] will be used to make a global analysis.

3. Linear growth rate

In this section, we will analyze the linear growth rate of tearing modes in the inner region of a singular layer at \( x = 0 \). In the inner region, \( \partial/\partial x \gg k \), so that equations (9) and (10) in the dominant order become

\[
\gamma \frac{\partial^2 \psi_1}{\partial x^2} = \frac{1}{2} k \mu_0 x \frac{\partial^2 \psi_1}{\partial x^2}, \quad (11)
\]

\[
\gamma \psi_1 + k \mu_0 x \phi_1 = \left( \eta + \frac{\alpha}{2} \right) \frac{\partial^2 \psi_1}{\partial x^2}, \quad (12)
\]

where \( \mu_0 = (\partial^2 \psi_0/\partial x^2)_{x=0} \). Then, the technique of reference [20] is used. Integrating equation (11), one can obtain

\[
\gamma \frac{\partial \psi_1}{\partial x} = \frac{1}{2} k \mu_0 \left( x \frac{\partial \psi_1}{\partial x} - \psi_1 + A \right), \quad (13)
\]

where \( A \) is the integral constant [20]. Introducing the transform \( \chi = x\partial \psi_1/\partial x - \psi_1 \), equation (12) can be derived as
is convenient to introduce the transform $x = \delta X$, then equation (14) can be written as

$$
\frac{\partial^2 \chi_1}{\partial X^2} - \frac{2}{X} \frac{\partial \chi_1}{\partial X} - (\omega + X^2) \chi_1 - A X^2 = 0,
$$

where

$$
\delta = 2^{1/4} (k \mu_0)^{-1/2} \gamma^{1/4} (\eta + \gamma d_e^2 / 2)^{1/4}, \quad \omega = 2^{2/3} \mu_0 \delta^2
$$

are defined for convenience. It can be found that the form of equation (15) is similar to that of the linear equation of resistive kink in the magnetohydrodynamics (MHD) frame [20, 23], only the form of $\omega$ is different. Consequently, the same method is used to solve equation (15).

Letting $s = X^2$, $\chi = \chi_0 / A$, equation (15) can be transformed to

$$
\frac{\partial^2 \chi}{\partial s^2} - \frac{1}{2 s} \frac{\partial \chi}{\partial s} - \frac{1}{4} \left(1 + \frac{\omega}{s}\right) \chi = -\frac{1}{4}.
$$

(16)

This equation can be solved by introducing the general Laguerre polynomials (one can see the details in references [20, 23]). Finally, the solution of equation (16) can be obtained as

$$
\chi_1 / A = -1 + 2^{-5/2} \omega \int_0^1 y^{\omega/4 - 1/4} (1 + y)^{1/2} \times \exp \left[ - \frac{1 - y}{2(1 + y)} X^2 \right].
$$

(17)

Now, in order to obtain the linear dispersion relation of tearing modes, the asymptotic matching between the solutions of the inner region and outer region is made, 

$$
\Delta' = \frac{1}{A} \int_0^{\lambda_H} \chi_1 (x) dx 
= \frac{1}{8} \int_0^{\infty} \chi_1 (x) dx
= \frac{\pi}{2} \frac{1}{\Delta} \Gamma(\omega/4 - 1/4)
= \frac{1}{8} \Delta \Gamma(\omega/4 + 5/4).
$$

(18)

Where, $\Delta' = \partial \phi \psi(0') / \partial x - \partial \phi \psi(0') / \partial x \phi \psi(0) = - \pi / \lambda_H$, where $\lambda_H$ is the ideal growth rate [2]. Next, based on equation (18), the curves representing the growth rate against the instability criterion $\Delta'$ are plotted for different values of $d_e$, where $\hat{d}_e = 2^{-2/3} (k \mu_0)^{2/3} \eta^{-1/3} d_e$, $\hat{\gamma} = 2^{2/3} (k \mu_0)^{-2/3} \eta^{-1/3} \gamma$ and $\Delta' = (k \mu_0)^{-1/3} \eta^{1/3} \Delta'$. are defined. In figure 1, the solid curve is plotted for $d_e = 0$, and the dashed and dotted curve are plotted for $d_e = 2$, $d_e = 4$, respectively. It can be seen that the growth rate increases as $d_e$ increases. It is obvious, since the effect of electron inertia is another mechanism of the magnetic field lines breaking in the generalized Ohm’s law, and it can also cause magnetic reconnection. It can also be seen that the enhanced efficiencies of the electron inertia are different for $1 / \Delta' \ll 1$ and $1 / \Delta' \gg 1$, which imply the drive from the free energy on tearing modes is large and small, respectively. Next, the tearing modes will be analyzed for these two cases.

First, when $1 / \Delta' \ll 1$, namely the drive on the tearing mode is large, it corresponds to $\omega \rightarrow 1$. Then the equation (18) can be reduced to

$$
\gamma^{3/2} = 2^{-1/2} k \mu_0 (\eta + \gamma d_e^2 / 2)^{1/2}.
$$

(19)

It is similar to the result of Porcelli [18] except for factors, since the problem is about pair plasmas. Equation (19) can be rewritten as

$$
\hat{\gamma}^{3/2} = (1 + \hat{\gamma} \hat{d}_e)^{1/2},
$$

(20)

where

$$
\hat{d}_e = 2^{-2/3} (k \mu_0)^{2/3} \eta^{-1/3} d_e, \quad \hat{\gamma} = 2^{2/3} (k \mu_0)^{-2/3} \eta^{-1/3} \gamma.
$$

If $\hat{d}_e \ll 1$, namely $d_e \ll \eta^{1/3}$, it corresponds to the effect of finite resistivity dominating over electron inertia, one can obtain

$$
\gamma \approx 2^{-1/3} (k \mu_0)^{2/3} \eta^{1/3} \left(1 + \frac{2^{1/3}}{6} (k \mu_0)^{2/3} \eta^{1/3} \hat{d}_e^2 \right)^{1/2},
$$

(21)

which is similar to the growth rate of the resistive kink mode, with only some modification due to the electron inertia effect.

If $\hat{d}_e \gg \eta^{1/3}$, the effect of electron inertia is dominant, the growth rate due to electron inertia can be obtained

$$
\gamma = \hat{\gamma} \left(1 + \frac{2 \eta \hat{d}_e^2}{k \mu_0 \hat{d}_e} \right)
$$

(22)

If $d_e \sim \eta^{1/3}$, the effects of resistivity and electron inertia are comparable, and neither can be neglected. From the above analysis, one can know that the dominance of either resistivity or electron inertia depends on the order of $d_e / \eta^{1/3}$. It can be seen explicitly in figure 2. In figure 2, the curve of the growth rate versus the electron inertia is plotted based on equation (20).
It can be seen that the growth rate behaves as equation (21) when $\hat{d}_{ek} \ll 1$, while it behaves as equation (22) when $\hat{d}_{ek} \gg 1$.

Second, when $1/\Delta' \gg 1$, namely the drive on tearing modes is small, it corresponds to the familiar constant-$\psi$ approximation. Equation (18) can be reduced to

$$\gamma_{fi}^{5/4} = (1 + \gamma_{fi} \hat{d}_{et}^2)^{5/4},$$

where

$$\gamma_{fi} = \frac{\gamma}{C_0^{4/5} \Delta^{4/5} \eta^{3/5}}, \quad \hat{d}_{et} = \frac{C_0^{2/5} \Delta^{2/5} d_e}{2^{1/2} \eta^{1/5}},$$

$$C_0 = 2^{-5/4} \frac{\Gamma(1/4)}{\Gamma(3/4)} (k \mu_0)^{1/2}.$$  

The same procedure is used as previously. If $\hat{d}_{et} \ll 1$, the resistivity dominates over the electron inertia, so that the growth rate can be obtained as

$$\gamma = C_0^{4/5} \Delta^{4/5} \eta^{3/5} \left(1 + \frac{3}{10} C_0^{4/5} \Delta^{4/5} \frac{d_e^2}{\eta^{2/5}}\right),$$

which is similar to the growth rate of resistive tearing mode in electron-ion plasma, with only some modification due to the electron inertia and different coefficients. This is obvious, since equations (7) and (8) are formally identical to resistive reduced MHD except for factors of $1/2$ in the limit $d_e \rightarrow 0$. If $d_e \gg \eta^{1/3}$, the growth rate can be derived as

$$\gamma = \left[\frac{\Gamma(1/4)}{4 \pi \Gamma(3/4)}\right]^2 k \mu_0 \Delta^2 d_e^3 \times \left[1 + \frac{3}{4} \frac{\Gamma(1/4)}{4 \pi \Gamma(3/4)} k \mu_0^{-1} \Delta \frac{\eta}{d_e^2}\right],$$

which is consistent with the in reference [24], with only some modification due to the resistivity. If $d_e \sim \eta^{1/3}$, the effects of the electron inertia and resistivity are both important. It can be seen explicitly in figure 3. It can be known that the effect of electron inertia dominates over resistivity when $d_e \gg \eta^{1/3}$ in this case. This is different from the first case, $1/\Delta' \ll 1$. In the first case, the effect of electron inertia dominates over resistivity if $d_e \gg \eta^{1/3}$.

### 4. Conclusions

The tearing modes due to the resistivity and electron inertia are investigated in pair plasmas. The unified linear dispersion of tearing modes is derived analytically, and is analyzed for $\Delta' \gg 1$ and $\Delta' \ll 1$. The scaling of the growth rate of tearing modes on the electron inertia is proposed.

The conditions under which resistivity or electron inertia dominates is different when $\Delta' \gg 1$ and $\Delta' \ll 1$. When $\Delta' \gg 1$, the resistivity dominates over electron inertia if $d_e \ll \eta^{1/3}$, while the electron inertia is dominant if $d_e \gg \eta^{1/3}$. The growth rate is $\gamma \propto \eta^{1/3} (1 + d_e^2 / \eta^{2/3})$ if the resistivity dominates over electron inertia, and $\gamma \propto d_e (1 + \eta / d_e^2)$ due to the electron inertia. When $\Delta' \ll 1$, corresponding to tearing mode instability, the effect of resistivity is dominant if $d_e \ll \eta^{1/3}$, while the electron inertia dominates if $d_e \gg \eta^{1/3}$. The growth rate is $\gamma \propto \eta^{1/3} (1 + \eta / d_e^2)$ due to the resistivity, and $\gamma \propto d_e^2 (1 + \eta / d_e^2)$ due to the electron inertia.

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