Letter

Geodesic acoustic mode in a reduced two-fluid model

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Abstract

A reduced two-fluid model is constructed to investigate the geodesic acoustic mode (GAM). The ion dynamics is sufficiently considered by including an anisotropic pressure tensor and inhibited heat flux vector, whose evolutions are determined by equations derived from the 16-momentum model. Electrons are supposed to obey the Boltzmann distribution responding to the electrostatic oscillation with near ion acoustic velocity. In the large safety factor limit, the GAM frequency is identical with the kinetic one to the order of $1/q^2$ when zeroing the anisotropy. For general anisotropy, the reduced two-fluid model generates the frequency agreeing well with the kinetic result with arbitrary electron temperature. The present simplified fluid model will be of great use and interest for young researchers and students devoted to plasma physics.

Keywords: geodesic acoustic mode, fluid model, tokamak

(Some figures may appear in colour only in the online journal)

The geodesic acoustic mode (GAM) [1] is a well-observed experimental phenomenon in tokamak plasmas. It is basically an electrostatic acoustic mode occurring on the magnetic surfaces with a radially local structure. The mode involves $m = 1$ pressure and density perturbations and $m = 0$ poloidal perturbed flow with toroidally symmetric spatial structure. It has attracted much attention during the last few decades (see, for example, [2–6] and references therein) due to its important role in modulating plasma turbulence. It is well known that a typical frequency of GAM performed in the ideal magnetohydrodynamic (MHD) model is $\omega_{\text{MHD}}^2 = \frac{5}{6}(1 + \tau)(2 + q^{-2})v_i^2/R^2$, where $R$ is the major radius of tokamaks, $q$ is the safety factor, $v_i = \sqrt{2T_i/m_i}$ is the ion thermal velocity and $\tau$ denotes $T_e/T_i$ with the ion mass $m_i$, electron and ion temperatures $T_e$ and $T_i$. On the other hand, the kinetic theory predicts $\omega_{\text{K}}^2 = (7/4 + \tau)v_i^2/R^2(1 + O(q^{-2}))$ [7, 8]. Apparently, there is a significant discrepancy between the MHD and kinetic results. It is very natural to ask: is it possible and how to eliminate this inconsistency? Researchers found that the two-fluid model could be able to give a solution to this question. Such a discrepancy has shown to be removed on the leading order by taking into account the anisotropic ion parallel viscosity in the two-fluid equations [9–12]. It was also discussed in detail by Storelli et al [13]. Besides, the reduced Braginskii framework has been adopted to study the stability of GAM [14].

After initially performed in the ideal MHD model in 1968 [1], the GAM has been intensively investigated by using the MHD equations, two-fluid model, and the kinetic theory. Unfortunately, the equations for the latter two models are quite laborious. Their mathematics is really involved in most times. Taking the two-fluid model as an example, it was expected that the fluid description could be briefer than the full kinetic theory. But previous work indicates that the two-fluid mathematical is still of great complication [9–12]. Besides, the current quasi-neutrality condition $\nabla \cdot \mathbf{j} = 0$ is additional used to derive the dispersion relation [12].

In the case of $\tau \ll 1$, the standard Chew-Goldberger-Low (CGL) model is shown to yield the GAM frequency identical with the kinetic one just on the leading order [15]. A discrepancy appears in the terms on the order of $O(1/q^2)$. The
flaw is fixed by using the fluid model derived from the 16-momentum equations [16]. The GAM frequency is then shown to be identical with the kinetic one to the order of 1/q^2 in the large q case when zeroing the anisotropy. Numerical evaluation also indicates that the 16-momentum result agrees well with the kinetic result for general q. However, the two works both focus on the ion thermal dynamics by assuming \( \tau \ll 1 \). Note that in a realistic tokamak plasma, there is \( \tau \gg 1 \) in the core and \( \tau \sim 1 \) at the edge of the plasma column in normal conditions. That means the electron temperature effect has to be taken into account in the dispersion relation of GAM as well as the collisionless damping of GAM [17]. This motivates the present work, which is devoted to combination of CGL-like equations and the effect of electron dynamics.

In the present work, we try to construct a reduced two-fluid model describing the low-frequency electrostatic perturbations with toroidally symmetrical structure by taking into account the electron thermal dynamics. The ion dynamic effect is fully considered by introducing an anisotropic pressure tensor and heat flux. Based on the reduced two-fluid model, we derive the GAM frequency without too involute algebraic manipulation, which can agree well with the kinetic result for arbitrary \( \tau \). When zeroing the anisotropy, the frequency is exactly identical with the kinetic one on the order of 1/q^2. Such a fluid model will be of great use and interest for young researchers and students devoted to plasma physics.

The basic equations of the reduced two-fluid are performed as follows. First, the ion momentum equation is obtained by adding the standard electron motion equation to the ion motion equations as follows. First, the ion momentum equation is obtained by young researchers and students devoted to plasma physics.

\[
\rho_i \frac{d\vec{v}_i}{dt} = -\nabla \cdot \vec{p}_i - \nabla p_e + \vec{J} \times \vec{B}.
\]  

(1)

Here, \( \rho_i = n_i m_i \) is the ion mass density, \( \vec{p}_i \) is the ion pressure tensor with a CGL form. Electrons are assumed to be isotropic with a scalar pressure \( p_e \). From the perpendicular component of electron motion equation, we still have the ideal frozen-in condition and then find

\[
\partial_t \vec{B} = \nabla \times (\vec{v}_i \times \vec{B}).
\]  

(2)

The continuous equation for ions is

\[
\partial_t p_i + \nabla \cdot (p_i \vec{v}_i) = 0.
\]  

(3)

Meanwhile, the parallel and perpendicular pressures of ions are determined by [18, 19]

\[
\frac{\rho_3}{B^2} \frac{d}{dt} \left( \frac{B^2 p_i}{\rho_i^3} \right) = -\nabla || q_i - (2q_e - q_i) \nabla || \ln B,
\]  

(4)

\[
\rho_i B^2 \frac{d}{dt} \left( \frac{p_i}{\rho_i B} \right) = -\nabla || q_i + 2q_e \nabla || \ln B,
\]  

(5)

respectively, in which \( \vec{q} = q_i \vec{B} + q_e \vec{B} \) is the inhibited heat flux, whose dynamic behavior is governed by [18, 19]

\[
\frac{d}{dt} \left( \frac{B^2 q_i}{\rho_i^3} \right) = -3\alpha \frac{p_i B^2}{\rho_i^3} \nabla \left( \frac{p_i}{\rho_i} \right),
\]  

(6)

A parameter \( \alpha \) is introduced additionally in equations (6) and (7) to identify the effects of heat conducting by following [16]. The standard CGL double adiabatic model is recovered when the equilibrium heat flux and heat conducting effect are both ignored by letting \( q_i = q_e = 0 \) and \( \alpha = 0 \). In the general non-CGL case, there is \( q_i \neq 0 \), \( q_e \neq 0 \), and \( \alpha \neq 1 \). For \( q_i = q_e = 0 \) and \( \alpha = 1 \), the equilibrium heat flux is supposed to be zero, but the perturbed heat flux is still taken into account, which is the case considered in the present work.

Electron dynamics is, in fact, not sufficiently considered in the equations above except a scalar pressure. To take into account electron dynamic effect, we need other equations determining the motion and state of electrons. It seems the classical one \( d(p_e \rho_e^{-1})/dt = 0 \) can be used. Then combining the electron continuous equation and motion equation encloses the model. The electrons perturbed density can be derived from those three equations. However, since the ideal frozen-in condition (2) is used, which is derived from the perpendicular electron motion equation, we can not use the full electron motion equation again to determine the electrons perturbed velocity. Of course, the current quasi-neutrality condition \( \nabla \cdot \vec{J} = 0 \) can be employed. But this will compound the derivation, eventually leading to the tiny difference between the present model and the standard two-fluid model. Considering that a typical frequency of the perturbation is about \( \omega \sim \nu_{Te}/R < \nu_{Te}/R \), where \( \nu_{Te} = \sqrt{2T_e/m_e} \) is the ion thermal velocity for two temperature Maxwellian distribution and \( \nu_{Te} \) is the electron thermal velocity, the parallel projection of electron momentum equation is reduced to

\[
0 = -T_e \nabla n_e + em_e \nabla \Phi.
\]  

(8)

Here, the fact that the perturbation is basically electrostatic for GAM is adopted. Strictly speaking, the electric field is \( \vec{E} = -\nabla \Phi - \frac{1}{4\pi} \partial_t \vec{A} \) in which \( \partial_t \vec{A} \) does not exactly equal zero but can be disregarded in the electron motion equation. As a result, electrons can reach the thermal equilibrium to maintain a Boltzmann distribution via quick response to the electric field since the perturbation velocity is much less than the electron thermal velocity. One then finds the electron density with an exponential dependence on \( \Phi \) as

\[
n_e = n_0 \langle \psi \rangle \exp \left( \frac{\phi}{T_e} \right).
\]  

(9)

Here, we write \( \Phi = \langle \psi \rangle + \phi \), where \( \langle \psi \rangle \) is the magnetic surface average and \( \phi \) is the part depending on the poloidal angle. Equations (1)–(7) and (9) comprise the reduced two-fluid model describing the electrostatic disturbance.

Now we linearize these equations to derive the dispersion relation of GAM. We consider a large-aspect-ratio tokamak plasma with toroidal symmetric magnetic fields and work in the \( (r, \theta, \zeta) \) coordinate system. The prefix \( \xi \) denotes the perturbed profile. The Lagrangian perturbation \( \xi \) is expanded into \( \xi \vec{B}/B + \xi_\theta \vec{B} \times \nabla \theta/B \), where the radial displacement is
neglected [20]. We obtain the poloidal and parallel projections of ion momentum equation as

\[-p_i\omega^2\xi_\theta + \frac{1}{\psi^3} g^{11}(\delta p_\perp + \delta p_\| + B\delta B_i) - \frac{B^2}{8\pi^2} \nabla^2_\perp \xi_\theta \]

\[-K_\theta(2B\delta B_i - \delta p_\perp + \delta p_\|) = 0, \quad (10)\]

\[-p_i\omega^2\xi_\| + \nabla_i(\delta p_\| + \delta p_\perp) = 0, \quad (11)\]

in which the perturbed pressures are governed by the perturbed entropy equations (4) and (5),

\[\delta p_\perp = -3p_\perp + \xi_\theta K_\theta g^{11} = \frac{1}{\omega^2} \nabla_\perp \delta q_i + \frac{1}{\omega^2} (2\delta q_i - \delta q_\|) \nabla_\perp \ln B, \quad (12)\]

\[\delta p_\| = -p_\| + \xi_\theta K_\theta g^{11} p_\| (\sigma - 3) + \frac{1}{\omega^2} \nabla_\| \delta q_\| - \frac{1}{\omega^2} (2\delta q_\| - \delta q_\perp) \nabla_\| \ln B. \quad (13)\]

According to equations (6) and (7), the perturbed heat fluxes are

\[\delta q_i = \alpha \frac{3p_\perp}{\omega^2} \nabla_\perp \delta p_\| + \frac{1}{\omega^2} \nabla_\| \delta p_\|, \quad (14)\]

\[\delta q_\| = \alpha \frac{p_\|}{\omega^2} \nabla_\| (\delta p_\perp + p_\|). \quad (15)\]

The four equations above can also be found in [16]. Here, \(\phi\) is short for \(\nabla \cdot \xi\). According to equation (10), in order to eliminate the fast magnetosonic wave, there must be \(\delta P = \delta p_\perp + \delta p_\| + B\delta B_i \approx 0\). And then, equation (10) can be simplified to

\[\langle -p_i\omega^2\xi_\theta \rangle + \langle K_\theta(\delta p_\perp + \delta p_\| + 2\delta p_\|) \rangle = 0. \quad (16)\]

In the low beta condition, \(\delta P = 0\) is reduced to \(\delta B_i = 0\), namely,

\[\phi + 2K_\theta\xi_\theta g^{11} = \frac{1}{\rho_i \omega^2} \nabla_\perp^2 (\delta p_\| + \delta p_\perp) = 0. \quad (17)\]

Generally, only the coupling effect between the zeroth order harmonic and the first sideband should be considered. We can assume \(\Phi = (\Phi) + \Phi_e \rho_e e^{i\theta}\). Recall that the kinetic theory shows \(\Phi_l = -\Phi_\perp\) [21]. Consequently, we can write \(\Phi = \Phi_l \sin \theta\) with only a sine component. A similar conclusion can be obtained from equation (10), the poloidal projection of the momentum equation, in which only the sine component of the perturbed pressure plays a role considering that \(K_\theta \approx \sin \theta / R\). Without loss of self-consistency, we can assume \(f = f_\perp \sin \theta\), where \(f\) denotes \(\delta p_\perp, \delta p_\|, \delta p_\|, \phi, \phi, \Phi, \Phi\). As a result, we have

\[\delta p_{\|,l} = -\frac{3 + 3\alpha}{1 - \frac{3\alpha^2}{2q^2}} p_\| \phi \Phi + \frac{\sigma - 3\alpha}{1 - \frac{3\alpha^2}{2q^2}} \frac{p_\|}{R} \xi_\theta, \quad (18)\]

\[\delta p_{\perp,\|} = -p_\| \sigma \Phi + \frac{\sigma(2\sigma - 1)}{1 - \frac{\alpha}{2q^2}} \frac{p_\|}{R} \xi_\theta. \quad (19)\]

The electron effect on the GAM dispersion relation is related to the electron perturbed pressure. Hence the last key point is to determine \(\delta p_e\). According to the ion continuous equation, there is \(\dot{n}_i = -n_0 \sigma.\) Using the charge quasi-neutrality condition \(\dot{n}_i = \dot{n}_e\), we find

\[\langle \dot{n}_i \rangle = 0, \quad (20)\]

\[\bar{\Phi}_e = \frac{T_e}{\epsilon} \phi_e. \quad (21)\]

Then, ignoring the temperature perturbation (this is reasonable since generally, the electron temperature relax rate is the fastest to maintain a constant temperature, leading to \(\gamma_e = 1\)), one has \(\delta p_e = -p_\| \tau \phi_e\) with \(\tau\) defined as \(T_e/T\), here, which is reduced to the classical definition in the isotropic limit.

Substituting \(\delta p_\perp, \delta p_\|, \delta p_\\|, \delta p_e\) into equations (16) and (17) to eliminate the perturbations yields the dispersion relation as

\[\Omega^2 - G_2^2 \Omega^4 + G_1^2 \Omega^2 - G_0 = 0, \quad (22)\]

in which

\[G_2 = \frac{3}{4} + \frac{\sigma}{2}(1 + \sigma) + \tau + \frac{1}{q^2} \left(\frac{3}{2} + \tau + 2\alpha\right), \]

\[G_1 = \frac{1}{8q^2} \left[\sigma^2(5 + 2\tau + \tau(3 - 2\sigma)) + \alpha(6 + 3\alpha + 4\tau)\right] + \frac{\alpha(6 + 3\alpha + 4\tau)}{4q^4}, \]

\[G_0 = \frac{\alpha}{16q^3} \left[\sigma^2(5 + 6\sigma) + 6\sigma(1 + \sigma + 2\tau + \tau(3 - 4\sigma))\right] + \frac{3\alpha^2}{8q^6}. \quad (23)\]

According to the previous discussion [16], we know that the cubic equation above has three roots. Two of them correspond to the low-frequency zonal flow modes and the third one is relative to the high-frequency GAM branch. Keeping terms to the order of \(1/q^2\) by neglecting higher-order terms, we obtain the GAM frequency as

\[\Omega^2 = \frac{3}{4} + \frac{\sigma}{2}(1 + \sigma) + \tau + \frac{1}{q^2} \left(\frac{2\tau + 3 + \sigma^2 + \alpha(3 + 2\tau + 2\sigma^2)}{6 + 4\sigma + 4\sigma^2 + 8\tau}\right). \quad (24)\]

For comparison, we rewrite the kinetic result (see equation (31) in [22]) for large safety factor here

\[\Omega_{K}^2 = \frac{3}{4} + \frac{\sigma}{2}(1 + \sigma) + \tau - \frac{\tau(1 - \sigma)^2}{4(1 + \tau)} + \frac{1}{q^2} \left(\frac{2\tau + 3 + \sigma^2 + 6 + \sigma^2}{6 + 4\sigma + 4\sigma^2 + 8\tau - \frac{2(1 - \sigma)^2}{1 + \tau}}\right). \quad (25)\]

As aforementioned, \(\alpha\) represents the heat flux effect. The standard CGL double adiabatic model can remove the discrepancy between the fluid model and kinetic theory just on the leading order in the case of \(\tau = 0\). While the presence of heat flux yields the GAM frequency identical with the kinetic one to the order of \(1/q^2\), in turn, making a further step to
remove the discrepancy. Hence here, we just focus on the case of $\alpha = 1$.

Tokamak plasmas are usually considered to be isotropic, even in the low-collisional or collisionless limit. When the auxiliary heating is not much intense, only small partition particles are heated to thermal particles or a few energetic particles are generated. The bulk plasma can still be isotropic. Thus, we take $\sigma = 1$ by ignoring the anisotropy in the equilibrium state. The difference between the entropy equations and the ideal state equation implies that although the equilibrium pressure is weakly anisotropic or just isotropic, the parallel and perpendicular pressure perturbations evolve differently due to the presence of strong magnetic field. They have different degrees of freedom. As a result, the GAM frequency is simplified to

$$\Omega^2 = \frac{7}{4} + \tau + \frac{1}{q^2} \frac{23 + 16\tau + 4\tau^2}{14 + 8\tau},$$

which is exactly identical with the previous kinetic result [21]. For a weakly anisotropic plasma with $\sigma \sim 1$, the fluid frequency agrees well with the kinetic result. Only slight discrepancy exists between the two results, as confirmed in figure 1, which shows that the fluid result is almost the same with the kinetic one when $\sigma$ is near unit. Only for $\sigma < 0.5$ or $\sigma > 1.5$, the fluid frequency is a little higher than the kinetic one.

From the aspect of experiments, the isotropic approximation of tokamak plasmas becomes quite unrealistic with intense auxiliary heating. For example, the plasma heating by the neutral beam injection (NBI), ion cyclotron resonance heating (ICRH), and electron cyclotron resonance heating (ECRH) can produce strong plasma anisotropy. Generally, the NBI (tangential injection) creates anisotropic equilibria with a larger energy parallel to the injection, while the ICRH heats the ions and typically drives perpendicular dominant anisotropy with $p_\perp > p_\parallel$ [23]. It is of interest to compare the fluid frequency with the kinetic result when $\sigma \neq 1$. Equation (24) is not exactly the same as the kinetic frequency (25). But the discrepancy between them is sufficiently ignorable. For instance, $\Omega_\text{f}$ is about 1.811 and $\Omega_\text{k}$ is about 1.809 for $\sigma = 1.2$, $\tau = 1$, and $q = 3.0$. To illustrate this point clearly, we numerically plotted the dependence of the GAM frequency determined by equation (22) and the kinetic result on the safety factor and temperature ratio $\tau$ in figures 2 and 3, respectively. Figure 2 shows that for small safety factor, i.e., $q < 2$, there is slight difference between the fluid result and the kinetic one. When $q$ is greater than 2, the fluid result agrees well with the kinetic frequency. Figure 3 confirms again that in the isotropic case, the GAM frequency derived from the reduced two-fluid model is almost the same as the kinetic result. For the cases of $\sigma = 0.5$ and $\sigma = 1.5$, there exists very slight discrepancy between the fluid and kinetic frequencies.

In conclusion, we used a reduced two-fluid model to derive the dispersion relation of GAM. The ion dynamic effect is adequately considered within the double entropy equations.
describing the anisotropic pressure and inhibited heat flux. In order to take into account the electron dynamic effect, we ignore the electron inertial term in the parallel projection of motion equation to obtain the electron density responding to the electrostatic oscillation. Then the charge quasi-neutrality condition is adopted to enclose the model and then derive the dispersion relation of GAM, as presented in equation (22). In the large safety factor limit, keeping terms to the order of 1/q^2 yields the simplified frequency of GAM, which is identical with the kinetic one by zeroing the anisotropy. For arbitrary temperature ratio and anisotropy, numerical evaluation shows that the fluid frequency derived from the reduced two-fluid model agrees very well with the kinetic result. Compared with the gyro-kinetic equation or the standard two-fluid model, the mathematical difficulty involved here has been dramatically reduced. It is expected that we can use such a fluid model to make further research on GAM, such as re-studying the toroidal rotation effect on the GAM. It is known that the dispersion of GAM in a toroidally rotating tokamak plasma derived in the ideal MHD equations is not the same as the one obtained from the gyro-kinetic equation [4, 6]. It is expected that the reduced two-fluid model can remove the discrepancy, which is the scope of our future work.

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