Nonlinear Simulations of Coalescence Instability Using a Flux Difference Splitting Method

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Abstract A flux difference splitting numerical scheme based on the finite volume method is applied to study ideal/resistive magnetohydrodynamics. The ideal/resistive MHD equations are cast as a set of hyperbolic conservation laws, and we develop a numerical capability to solve the weak solutions of these hyperbolic conservation laws by combining a multi-state Harten-Lax-Van Leer approximate Riemann solver with the hyperbolic divergence cleaning technique, high order shock-capturing reconstruction schemes, and a third order total variance diminishing Runge-Kutta time evolving scheme. The developed simulation code is applied to study the long time nonlinear evolution of the coalescence instability. It is verified that small structures in the instability oscillate with time and then merge into medium structures in a coherent manner. The medium structures then evolve and merge into large structures, and this trend continues through all scale-lengths. The physics of this interesting nonlinear dynamics is numerically analyzed.

Keywords: magnetohydrodynamics, nonlinear simulation, finite volume method, coalescence instability

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(Some figures may appear in colour only in the online journal)

1 Introduction

Numerical simulation of magnetohydrodynamics (MHD) has become one of the most important methods in plasma physics research. Many successful numerical tools, such as the NIMROD code [1] and the M3D code [2] have been developed and applied. For most of the simulation tools that have been widely used, the traditional operator discretization methods are adopted as the main algorithm.

To meet the demand of long-term accuracy and fidelity for simulation studies of multi-scale MHD dynamics such as magnetic reconnections and tokamak disruptions, numerical algorithms that are able to capture discontinuities and to satisfy exact conversation laws are needed. One of these algorithms is the Godunov type method or the flux difference splitting (FDS) method for solving time dependent hyperbolic conservation laws [3]. It has gained great success and becomes a very important branch in computational fluid dynamics (CFD). In practice, this kind of numerical approach has been applied in many research fields, especially in aviation design, aerospace engineering, hydraulics, weather forecast, and turbulence research, due to its high resolution of discontinuities and satisfactory accuracy.

In the present study, we have developed a flux difference splitting MHD simulation capability based on finite volume spatial discretization. With application of a high resolution Harten-Lax-Vanleer-Discontinuities (HLLD) [4] Riemann solver, we can solve strong nonlinear structures like a very steep magnetic field (or current sheet) sharply without suffering too much error from numerical diffusivity. High temporal accuracy and robustness can be obtained by using the strong stability preserving 3rd order total variation diminishing (TVD) Runge-Kutta (RK) time intergration method. As a numerical example, we have applied the developed simulation code to study the nonlinear evolution of the coalescence instability. It is found that the nonlinear evolution of coalescence instability has a coherent space-time structure. Small structures oscillate in time and result in magnetic discontinuities, which then reconnect quickly to form medium structures. These medium structures continue to evolve into large structures, and the dynamics...
show a self-similar pattern. The exact conservation properties and discontinuity-capturing capability play an indispensable role in obtaining these crucial physics results.

This paper is organized as follows. In section 2, we will introduce the numerical scheme of the ideal/resistive MHD equations based on finite volume discretization. An HDC technique is introduced to overcome the difficulty of numerical divergence controlling. In section 3, we will use the developed simulation code to study the nonlinear evolution of the coalescence instability, and demonstrate the capabilities and advantages of this method.

2 Numerical approach

The FDS method, or so called Godunov [3] method, solves the weak solutions of the integrated form of hyperbolic conservation laws. It can be taken as a 3-step method. Firstly, conserved variables are discretized, and reconstruction schemes [5-7], such as the Monotonic Upstream-Centered Scheme for the Conservation Laws (MUSCL) method, can be applied for high order spatial accuracy. For multi-dimensional cases, only a rotation transform in the norm direction of the corresponding cell interface is needed to form a local 1D Riemann problem [8]. Secondly, these constructed local Riemann problems are solved by some approximate Riemann solvers such as the Roe or Harten-Lax-van Leer-Contact (HLLC) solvers [9-11] for numerical fluxes on all cell interfaces. Thirdly, the time evolution is solved by an explicit or implicit method. For general hyperbolic conservative systems, the schemes used for first and third steps can be all the same, thus in this methodology the physical part and geometry part can be divided conveniently. In the second step, depending on the physical models we need to select different methods for solving the Riemann problems exactly or approximately.

We start from an MHD system in mass, momentum and energy conservative form. If we assume the typical density, magnetic field and space scale are of order $\rho_0$, $B_0$ and $L_0$ respectively, then we can use these constants to normalize the relevant physical variables and get the conservative form of dimentionless MHD equations as

$$\partial_t \rho + \nabla \cdot (\rho \mathbf{V}) = 0, \quad (1)$$

$$\partial_t (\rho \mathbf{V}) + \nabla \cdot \left[ \rho \mathbf{V} \mathbf{V} - \mathbf{B} \mathbf{B} + \left( p + \frac{1}{2} B^2 \right) \mathbf{I} \right] = 0, \quad (2)$$

$$\partial_t \left[ \frac{p}{\gamma - 1} + \frac{1}{2} (\rho \mathbf{V}^2 + B^2) \right] + \nabla \cdot \left[ \frac{\gamma p}{\gamma - 1} + \frac{1}{2} \rho \mathbf{V}^2 + B^2 \right] \mathbf{V} - (\mathbf{B} \cdot \mathbf{V}) \mathbf{B} + \eta J \times \mathbf{B} = 0, \quad (3)$$

$$\partial_t \mathbf{B} + \nabla \cdot [(\mathbf{BV}) - (\mathbf{VB}) - \eta \nabla \mathbf{B}] = 0, \quad (4)$$

$$\nabla \cdot \mathbf{B} = 0. \quad (5)$$

In which the pressure is normalized by $B_0^2/\rho_0$, the speed is normalized by Alfven speed $V_A = \sqrt{B_0^2/\mu_0 \rho_0}$, and the time is normalized by $L_0/V_A$. $\eta$ is resistivity and the current density $J$ is obtained by Ampere’s law. For numerical approach designing, we will first project this set of equations into 1D space in the $x$-direction in the absence of resistivity to discuss the Riemann solver which is supposed to be used to calculate numerical fluxes on cell interfaces. In this case, the spatial differential operator $\nabla$ in Eqs. (1)-(5) is replaced by $\partial/\partial x$, and the field of $B_x$ is constant due to Eq. (5). It is straightforward to show that there are seven distinct eigenvectors and eigenvalues $V_s$, $V_x \pm c_s$, $V_x \pm c_a$, and $V_x \pm c_t$. Here, $c_s$, $c_a$, and $c_t$ are the velocities for slow wave, Alfven wave and fast wave, respectively,

$$c_s = \sqrt{\frac{\gamma p + B^2 - \sqrt{\gamma p + B^2}^2 - 4 \gamma p B_x^2}{2 \rho}}, \quad (6)$$

$$c_a = \frac{B_x}{\sqrt{\rho}}, \quad (7)$$

$$c_t = \sqrt{\frac{\gamma p + B^2 + \sqrt{\gamma p + B^2}^2 - 4 \gamma p B_x^2}{2 \rho}}. \quad (8)$$

The exact Riemann solver for the above MHD system is too complicated and time consuming to apply in a practical numerical code, and an approximate Riemann solver is required. The first attempt of an approximate Riemann solver for the MHD system was made by Brio and Wu [12] for the simplified case of $\gamma = 2$ by applying an average state of the Roe type. A practical linear Roe solver was later proposed by Cargo and Gallice [13], and Balsara [14]. This linear approximate solver has become a standard for its high resolution and accuracy. However, it is still not quite efficient and the problem of negative density and pressure remains.

Learning from the success story of nonlinear HLL [15] or HLLC type approximate Riemann solvers for the Euler system, several HLL type of Riemann solvers have been developed [4,16-18]. Even though these methods can maintain positive density and pressure, most of them cannot capture all the discontinuities associated with the seven eigenvalues. In the present study, we adopt a multi-state HLL [a.k.a. Harten-Lax-Van Leer-Discontinuities (HLLD) [4]] Riemann solver, which assumes six states with five discontinuities in the Riemann fan. Judging from the result of the 1D shock tube test with first order scheme and a single Euler forward method, this solver performs as well as Roe type solvers, even with discontinuities associated with slow waves, which was not assumed in the multi-state approximation. That is to say, this HLLD solver has a minimum level of numerical diffusivity among those solvers thus will not smooth structures like discontinuities too much, while at a reasonable computation cost. Some multi-dimensional Riemann solvers [19] have also been developed very recently.
However, they are too complicated and far from being practically usable. Currently, the HLLD solver for the classical 1D Riemann problem is our first choice.

In the 1D case, the Riemann solver will not generate a flux corresponding to the constant \( B_x \). However, in multi-dimensions the divergence of \( B \) may become non-zero due to numerical error, which sometimes can even grow with time and lead to the collapsing of the calculation. Numerical error of magnetic field divergence must be controlled. There are several methods to do so, including Powell’s eight-wave method \([20,21]\), the constrained transport (CT) method, projection method, and the hyperbolic divergence cleaning (HDC) method. In Powell’s eight-wave formulation, additional source terms proportional to the magnetic field divergence are added into the equations, and Eq. (5) is abandoned. This scheme requires a simple modification of the basic Riemann solver; however, its non-conservative nature will lead to an incorrect jump at strong discontinuities. The CT method works well except that it can only be easily implemented on Cartesian or curvilinear grids. The projection method requires a Poisson solver that needs global inversion, which is very expansive for 3D calculations. Here, we adopt the HDC method by Dender et al. \([22]\), which introduces a general Lagrangian multiplier (GLM) \( \psi \) in the MHD formulation to form a hyperbolic GLM-MHD system. The equations of \( B_x \) and \( \psi \) are decoupled from other parts of the system, and after reconstruction the magnetic field \( B_x \) and \( \psi \) at an interface are \([22]\)

\[
\begin{pmatrix}
B_{x,m} \\
\psi_m
\end{pmatrix} = \begin{pmatrix}
B_{x,1} \\
\psi_1
\end{pmatrix} + \begin{pmatrix}
\frac{1}{2} (B_{x,r} - B_{x,l}) - \frac{1}{2\epsilon m} (\psi_r - \psi_l) \\
\frac{1}{2} (\psi_r - \psi_l) - \frac{1}{\epsilon} (B_{x,r} - B_{x,l})
\end{pmatrix}.
\tag{9}
\]

In Eq. (9), the subscripts \( m \), \( l \) and \( r \) denote intermediate, left and right values at an interface, and \( \epsilon \) is the global maximum velocity, usually taken to be \( \sqrt{V_g^2 + c}\). The corresponding flux can be calculated before the value of \( B_x \) is substituted into an arbitrary Riemann solver, and the total flux for all components and \( \psi \) are given subsequently.

To summarize, the following numerical schemes are used in our studies. For flux calculations we use the HLLD Riemann solver to obtain the convective part of the total flux, and the central scheme for diffusive flux induced by resistivity. For spatial reconstruction we adopt a second order shock capturing scheme of the MUSCL type, and for divergence controlling we apply the HDC method. After the fluxes for all components are calculated, we use a third order TVD-RK method \([23]\) to advance all conservative variables in time. We combine all the techniques mentioned above to form a complete capability of simulating nonlinear MHD equations in 2D geometry. Validation and verification of our numerical approach have been carried out with numerous examples, such as 1D shock tubes and the 2D Orszag-Tang vortex.

### 3 Coalescence instability

In this section, we apply the numerical tool developed to study the nonlinear evolution of the coalescence instability \([24–26]\). The equilibrium under investigation is a two-dimensional array of rectangle magnetic islands. This equilibrium is unstable with respect to a small initial perturbation, and will evolve into an intermediate pentagon structure and later into a state with current sheets or magnetic field discontinuities. Strauss and Longcope \([24,25]\) studied the coalescence instability analytically and numerically using a reduced ideal MHD system without energy conservation law. In their numerical simulation, a finite element method based on an adaptive refinement triangular mesh is used. The refinement happens when the gradient in the normal direction of the magnetic field increases, and one triangular element will be split into two if the product of current density times triangle area exceeds a threshold. In the simulation, a very thin current sheet is formed in the central region of the simulation box \([24]\), and the resistivity will drive magnetic reconnection and result in the merging of magnetic islands. However, the adaptive refinement algorithm generates very small grids during the formation of the current sheet which requires an extremely small time-step to satisfy the CFL condition for the explicit leap frog time advancing scheme. The calculation becomes too time consuming. Since the model does not contain a pressure and energy equation, the dynamics studied, especially the energy transfer process, was not complete. In a recent study by Zhou et al. \([26]\), an advanced algorithm is developed by discretizing Newcomb’s Lagrangian for ideal magnetohydrodynamics (MHD) in the Lagrangian labeling using discrete exterior calculus. The unique advantage of this new algorithm is that it is free from any numerical reconnection, and therefore specially valuable for simulating the coalescence instability where current sheet formation plays an important role. Indeed, simulation results showed that there is no magnetic reconnection caused by numerical errors and magnetic islands do not merge. However, with arbitrary small resistivity the magnetic reconnection can happen and the time evolution of this instability is quite different, so the nonlinear evolution of such instability is worth re-investigating.

In the nonlinear evolution of MHD, discontinuity-like structures often appear from smooth initial states and the ability of shock capturing in the nonlinear simulation is always required. In the present study, the initial smooth equilibrium with a small kinetic perturbation will evolve into an intermediate state with a very thin, but high density current sheet, so the tangential magnetic field can vary strongly in very few cells in the normal direction. The flux difference
splitting method applied by us is suitable for studying the coalescence instability for the following reasons. First, the weak solution of a hyperbolic conservation law is in general piecewise smooth, which means non-smooth profiles, or large differences between adjacent cells are allowed, and solved robustly. The numerical diffusivity may pollute the solution, so the application of a high resolution HLLD Riemann solver can reduce this bad effect better than any other solvers. The other reason is that the energy dynamics is crucially important for the nonlinear evolution, and this has not yet been systematically investigated. Since our numerical methods maintain all the conservation laws, it can be used to develop a much more comprehensive understanding of the nonlinear coalescence instability.

The simulation area is $[-0.5, 0.5]^2$ with a $512 \times 512$ grid and periodical boundaries. Initial conditions are $A = 0$, it can be used to develop a much more comprehensive numerical methods maintain all the conservation laws, which means non-smooth profiles, or large differences between adjacent cells are allowed, and solved robustly. The numerical diffusivity may pollute the solution, so the application of a high resolution HLLD Riemann solver can reduce this bad effect better than any other solvers. The other reason is that the energy dynamics is crucially important for the nonlinear evolution, and this has not yet been systematically investigated. Since our numerical methods maintain all the conservation laws, it can be used to develop a much more comprehensive understanding of the nonlinear coalescence instability.

The simulation area is $[-0.5, 0.5]^2$ with a $512 \times 512$ grid and periodical boundaries. Initial conditions are $\rho = 1$, $p = 1 + 8\pi^2 A^2$, and $B = \nabla A \times e_z$, where $A = 0.05[\cos(4\pi x) - \cos(4\pi y)]$ is the magnetic potential. A very small resistivity $\eta = 1.0 \times 10^{-6}$ is used. In Fig. 1(a), colored contours of the initial magnetic potential $A$ are shown. The magnetic field lines lay on the contours, and on red islands the field lines are counterclockwise while on blue islands they are clockwise.

![Fig. 1](image_url) Contour lines of magnetic potential at different time slices

The time evolution curve of the total kinetic energy $E_K$ is shown in Fig. 2 in the logarithm scale for $0 \leq t \leq 4.0$. It is evident that from about $t = 0.7$ to $t = 2.0$, the dynamics is in the linear phase, and the linear growth rate is measured to be $\gamma = 2.0$. Since the resistivity is so small, the resistive effect comparing to simulations with zero resistivity or some other very small value can be barely investigated. At $t = 2.8$, the kinetic energy reaches a local maximum, when magnetic energy reaches a local minimum. The contour of the magnetic potential at this time slice is shown as Fig. 1(b), where a current sheet or a “cliff” of the tangential magnetic field are formed. Plotted in Fig. 3 is the horizontal magnetic field $B_x$ along the $y$ axis at $t = 2.8$ and $x = 0$. A jump is shown at $y = 0$, indicating the position of a thin, high current density region (current sheet) here. These results in linear and nonlinear phases agree with those in Ref. [24]. As mentioned previously, at this phase of simulation, the adaptive refinement algorithm used in Ref. [24] will generate very small grids, which requires extremely small time-steps and larger computing power to continue. On the other hand, the high resolution shock capturing method that we adopted is able to handle the steep structure of magnetic field in cell-size resolution without having been smoothed out. Meanwhile there is no need to refine the mesh or reduce the time-step.

![Fig. 2](image_url) Evolution of total kinetic energy from $t = 0.0$ to $t = 4.0$. A linear fitting is also shown

![Fig. 3](image_url) Discontinuity in $B_x$ along the $y$ axis at $x = 0$ and $t = 2.8$

![Fig. 4](image_url) Contour plot of magnetic potential at $t = 3.2$. A small island emerges in the center of the current sheet

As our simulation carries on through the formation of a current sheet, the magnetic field starts to reconnect and a small magnetic island emerges in the center of the current sheet, as shown in Fig. 4. Apparently the dynamic hereafter is not consistent with the ideal MHD, since magnetic reconnection is not allowed to happen and the gradient of the tangential magnetic field is growing toward infinity. Thus some resistivity must be introduced. Here in our simulation the reconnection is caused by this finite but very small resistivity, so an
magnetic reconnection. At MHD time scale, small islands start to merge due to evolution of current sheets. Resistivity is necessary and important at the nonlinear to us the question of why the existence of a so small is the moving speed of the discontinuity. This implied a discontinuous, artificial dissipation term to the hyperbolic system such that the system becomes uniquely determined and the solution can be solved, then a physically relevant solution e.g. the “entropy solution” is defined as the limit of the solution when this add-on dissipation approaches zero. In order to find an appropriate approximation of the entropy solution numerically, discontinuities of admissible solutions need to satisfy the entropy condition

\[ \frac{f(U) - f(U_L)}{U - U_L} \geq S \geq \frac{f(U) - f(U_R)}{U - U_R}, \]

for all \( U \) between \( U_L \) and \( U_R \) for each of the conservation laws. Here, \( U \) stands for conserved quantities as mass, momentum and energy, the subscripts L and R stand for left and right quantities of a discontinuity, \( f(U) \) is flux corresponding to \( U \), and \( S \) is the moving speed of the discontinuity. This implied to us the question of why the existence of a so small resistivity is necessary and important at the nonlinear evolution of current sheets.

While oscillating back and forth with a typical MHD time scale, small islands start to merge due to magnetic reconnection. At \( t = 11.5 \), the intermediate state with pentagon structure collapses into an array of larger islands, whose orientation also rotates by 45 degrees relative to the initial small islands, as shown in Fig. 1(c). Unsurprisingly, this state also oscillates and is not stable. It continues to merge into larger islands until reaching a structure comparable to the simulation domain such merger stops numerically. During the long time evolution, the ability of shock capturing is frequently needed to calculate the discontinuities in magnetic field while the system is evolving towards larger structures. The near final state at \( t = 30.0 \) is shown in Fig. 1(d).

The time evolutions of kinetic energy, magnetic energy and internal energy are plotted in Fig. 5. The total energy is conserved, as expected. In the case studied here, there are two phase transitions for the dynamics at \( 7.0 < t < 8.0 \) and \( 12.0 < t < 15.0 \), corresponding to two fast magnetic island merging events. During each phase transition, smaller islands merge, and kinetic energy increases while magnetic energy drops quickly. Most of the loss in magnetic energy goes to thermal energy, and only a very small fraction is converted to kinetic energy. Obviously, it is the magnetic energy that provides the free energy and drives the instability. From the history of magnetic energy, we also observe that during each phase transition, half of the magnetic energy is released. These interesting features of the dynamics are revealed in the simulation because of the exact conservation properties of the algorithms adopted.

![Figure 5](image_url) Time evolution of energy from beginning to \( t = 30 \)

To further demonstrate the series of phase transitions, we now start the simulation from a smaller initial equilibrium array. As shown in Fig. 6(a), the scale length of initial magnetic islands and the period of perturbation momentum are reduced by a half relative to that in Fig. 1(a). As expected, we also observe the merging of smaller islands into larger ones in Fig. 6. This trend continues to a final state at \( t = 35 \) in Fig. 6(d) when the further island merging is restricted by the size of the simulation domain. This can be easily understood from the viewpoint in terms of energy. The
magnetic energy is the free energy source driving the instability, and the next larger structure has only half the magnetic energy. Therefore, the system will always evolve towards the larger structure with lower magnetic energy.

![Fig.6 Contour lines of magnetic potential at different time slices from a refined initial equilibrium](image)

### 4 Conclusion

We have developed a Flux Difference Splitting (FDS) simulation capability for MHD systems based on a finite volume spatial discretization. Especially, the recently developed Harten-Lax-Van Leer-Discontinuities Riemann solver and the Hyperbolic Divergence Cleaning (HDC) method are applied with a 3rd order Total Variance Diminishing (TVD) Runge-Kutta time evolving scheme. This numerical approach solves for the weak solution of the integrated form of ideal/resistive MHD equations in the forms of conservation laws for mass, momentum, energy, and magnetic flux. Using these schemes with high resolutions, we are able to solve nonlinear discontinuity-like structures as the current sheet accurately and obtain physical relevant solutions. The algorithm has been implemented and tested in a 2D unstructured grid, and applied to study the long time nonlinear dynamics of the coalescence instability in which the capabilities of maintaining conservation laws, capturing discontinuities are required. It is verified that small structures in the instability oscillate with time and then merge into medium structures in a coherent manner. The medium structures then evolve and merge into larger structures, and this trend continues through all scale-lengths. The energy curves simulated show the time evolution of magnetic, kinetic and internal energy while keeping the total energy conservative.

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