Damping Solitary Wave in a Three-Dimensional Rectangular Geometry Plasma

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Abstract The solitary waves of a viscous plasma confined in a cuboid under the three types of boundary condition are theoretically investigated in the present paper. By introducing a three-dimensional rectangular geometry and employing the reductive perturbation theory, a quasi-KdV equation is derived in the viscous plasma and a damping solitary wave is obtained. It is found that the damping rate increases as the viscosity coefficient increases, or increases as the length and width of the rectangle decrease, for all kinds of boundary condition. Nevertheless, the magnitude of the damping rate is dominated by the types of boundary condition. We thus observe the existence of a damping solitary wave from the fact that its amplitude disappears rapidly for $a \to 0$ and $b \to 0$, or $\nu' \to +\infty$.

Keywords: damping solitary wave, viscous plasma, reductive perturbation theory, quasi-KdV equation

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(Some figures may appear in colour only in the online journal)

1 Introduction

In recent years, a great deal of interest has been devoted to the application of plasma, such as the stability and instability of the plasma [1], fast ignition fusion, controlled fusion [2,3] and space weather [4]. In this literature, the study of linear as well as nonlinear wave phenomena in plasmas [5–7] attracted much attention due to their frequent occurrence in the pulsar magnetosphere [8,9], solar atmosphere [10], electron beams [11], active galactic nuclei [12] etc. Among the numerous studies, the nonlinear wave theories are more promising, since this kind of wave may arise from the instability of a certain class of initial conditions. One of the most promising nonlinear waves is the solitary wave [13–18]. First observed in water waves by Russell in 1834, the unchanged propagating wave was named the solitary wave. The nonlinear evolution of such small-amplitude waves is described by the well known Korteweg–de Vries (KdV) equation [19–21], which is extensively studied in many other branches of physics nowadays. The existence of solitary waves has been proved in many different types of plasma model due to a delicate balance between nonlinearity and dispersion, and the results are consistent with the theory and experimental findings. Haragus and Adlam presented a theoretical study of a solitary wave in cold plasma [22,23]; the former studied small amplitude in one spatial dimension while the latter focused on large-amplitude nonlinear hydromagnetic waves. Hazeltine and Mehdipour investigated solitary waves in magnetized plasma [24,25], while Lee and Sahu et al. investigated ion-acoustic and electron-acoustic solitary waves in a relativistic plasma system [26–28].

As aforementioned, both the linear and the nonlinear waves in plasmas are related to the fusion system; all of these wave modes on magnetic confined fusion may be stable or unstable, which can determine the realization of the fusion. Furthermore, the plasma is usually confined in a finite geometry both in a confined fusion device and in an accelerated driven system [29,30]. It is observed that nearly all of the previous studies of the waves in plasma focused on plasma in infinite regions, while it is almost impossible to find an unbounded plasma in the laboratory. The plasmas are all confined in a pipe or a rectangle in magnetic confined

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fusion and accelerated driven system devices, which are closer to the reality. Recently, we investigated the propagation of solitary waves in a bounded dust plasma together with a viscosity bounded plasma. Some plasma waves in the geometrically confined plasma have been studied, such as the round confined waves, but the others still remain unsolved. By introducing a rectangular geometry and employing the reductive perturbation theory, the present paper will discuss wave modes confined in the rectangular geometry, i.e., the plasma is limited to within \( x = 0, x = a \) and \( y = 0, y = b \) planes. The remaining parts of the paper proceed as follows. The derivation of a quasi-KdV equation in rectangular geometry is presented in section 2. We study the specific expression of damping rate under three types of boundary condition in sections 3, 4 and 5, respectively. The discussion and conclusion are given in section 6.

## 2 KdV type equation in a rectangular geometry

We consider a three-dimensional rectangular geometry plasma which is homogeneous, collisionless and unmagnetized. We also assume specific planes \((x = 0, x = a \) and \( y = 0, y = b \)) as the boundaries, and the plasma only moves along the \( z \) direction. Based on these hypotheses, the dynamics of solitary waves in such a plasma is governed by the following well known generalized hydrodynamic equations:

\[
\frac{\partial n_i}{\partial t} + \frac{\partial(n_i V)}{\partial z} = 0, \tag{1}
\]

\[
\frac{\partial V}{\partial t} + V\frac{\partial V}{\partial z} - \nu \frac{\partial^2 V}{\partial x^2} + \frac{e}{m_i} \frac{\partial \phi}{\partial z} = 0, \tag{2}
\]

where \( n_i \) is the ion number density, \( x, y \) and \( z \) are the space coordinates, \( e \) is the unit electron charge (absolute value \( 1.602 \times 10^{-19} \text{ C} \)), \( t \) is the time variable, \( m_i \) is the mass of the ions, \( \nu \) is the kinematic viscosity coefficient, \( V \) is the ion speed vector, and \( \phi \) is the electrostatic potential of the plasma.

If we further assume that the plasma is confined in a rectangular cube, it can flow only in the \( z \) direction. Hence, the Poisson equation in the three-dimensional rectangular coordinates can be simplified as

\[
\frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} + \frac{\partial^2 \phi}{\partial z^2} = 4\pi e(n_e - n_i), \tag{3}
\]

where \( n_e \) is the electron number density, which satisfies the Boltzmann distribution \( n_e = n_{e0} \exp(\frac{\psi - \phi}{k_B T_e}) \), \( k_B \) is the Boltzmann constant, \( T_e \) is the electron temperature, and \( n_{e0} \) is the unperturbed electron density.

We then normalize all the physical quantities. The electron density \( n_e \) is normalized by \( n_{e0} \), the electrostatic potential \( \phi \) is normalized by \( \frac{k_B T_e}{e} \), the velocity \( V \) is normalized by \( \frac{k_B T_e}{m_i} \), the space coordinates \( x, y, z \) and \( t \) are normalized by the electron Debye length \( \lambda = (\frac{k_B T_e}{4\pi e n_{e0}})^{1/2} \) and the plasma frequency \( \omega_{pi}^2 = (\frac{4\pi e^2 n_{i0}}{m_i})^{-1/2} \), and \( \nu \) is normalized by \( \lambda^2 \omega_{pi}^{-1} \) (here, \( \omega_{pi} = 2\pi f \), \( f \) is the plasma frequency). Therefore, we have the normalized equations in the form

\[
\frac{\partial n_i}{\partial t} + \frac{\partial(n_i V)}{\partial z} = 0, \tag{4}
\]

\[
\frac{\partial V}{\partial t} + V\frac{\partial V}{\partial z} - \frac{\nu}{m_i} \frac{\partial^2 V}{\partial x^2} + \frac{e}{m_i} \frac{\partial \phi}{\partial z} = 0, \tag{5}
\]

\[
\frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} + \frac{\partial^2 \phi}{\partial z^2} = n_e - n_i, \tag{6}
\]

\[
n_e = \exp(\phi). \tag{7}
\]

At equilibrium, a particular solution is easy to find for \( \pi_1 = 1, \pi_2 = 0 \) and \( V = 0 \). We give a perturbation to the solution, i.e., \( n_i = 1 + \delta n, \phi = \delta \phi, V = \delta V \) (here \( |\delta n| << 1, |\delta \phi| << 1, |\delta V| << 1 \)), and linearize this spatially homogeneous state,

\[
\frac{\partial \delta n}{\partial t} + \frac{\partial \delta V}{\partial z} = 0, \tag{8}
\]

\[
\frac{\partial \delta V}{\partial t} + \frac{e}{m_i} \frac{\partial \delta \phi}{\partial z} = 0, \tag{9}
\]

\[
\frac{\partial^2 \delta \phi}{\partial x^2} + \frac{\partial^2 \delta \phi}{\partial y^2} + \frac{\partial^2 \delta \phi}{\partial z^2} = \frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial z^2}. \tag{10}
\]

We then assume \( \delta \phi = X(x)Y(y)\psi(z,t) \) and substitute it into Eq. (11); we have

\[
\frac{\partial^2 X}{\partial x^2} + \frac{\partial^2 Y}{\partial y^2} - \frac{\partial^2 \psi}{\partial z^2} - \frac{\partial \psi}{\partial z} = -p. \tag{12}
\]

Here Eq. (12) can be rewritten as the following two equations:

\[
\frac{\partial^2 X}{\partial x^2} + \frac{\partial^2 Y}{\partial y^2} + p = 0, \tag{13}
\]

\[
(1 + p)\frac{\partial^2 \psi}{\partial z^2} = \frac{\partial^2 \psi}{\partial z^2} = 0, \tag{14}
\]

where \( p \) is a constant. From Eq. (13), we obtain

\[
\frac{\partial^2 X}{\partial x^2} - \lambda_0 X = 0, \tag{15}
\]

\[
\frac{\partial^2 Y}{\partial y^2} - \beta_0 Y = 0, \tag{16}
\]

where \( \lambda_0 \) and \( \beta_0 \) are both constants, and \( \lambda_0 + \beta_0 + p = 0 \). Substituting \( X(0) = X(a) = 0 \) and \( Y(0) = Y(b) = 0 \) into Eqs. (15) and (16), we get

\[
X(x) = \sum_{n=1}^{\infty} D_n \sin(\frac{n\pi x}{a}), \tag{17}
\]
With sinusoidal traveling-wave solutions $\psi^{(kz-\omega t)}$ and Eq. (11), we obtain the dispersion relation:

$$\omega^2 = \frac{k^2}{1+\kappa^2}. \quad (19)$$

Now consider a general case. From the above expressions it is clear that the $x$ and $y$ components are uncoupled from the $z$ and $t$ components. Thus, we can assume that the physical quantities $n_1$, $\varphi$ and $V$ have the following forms:

$$n_1(x, y, z, t) = X(x)Y(y)N(z, t) + 1, \quad (20)$$

$$V(x, y, z, t) = X(x)Y(y)u(z, t), \quad (21)$$

$$\varphi(x, y, z, t) = X(x)Y(y)\psi(z, t). \quad (22)$$

Substituting Eqs. (20)-(22) into Eq. (6), we have the following relationship:

$$\frac{\partial^2 X}{\partial x^2} + \frac{\partial^2 Y}{\partial y^2} = \frac{\psi - N_1 - \frac{\partial^2 \psi}{\partial x^2}}{\psi} = -q, \quad (23)$$

or, equivalently

$$\frac{\partial^2 X}{\partial x^2} + \frac{\partial^2 Y}{\partial y^2} + q = 0, \quad (24)$$

$$\psi - N_1 - \frac{\partial^2 \psi}{\partial x^2} + q\psi = 0, \quad (25)$$

where $q$ is a constant. Since each item is independent in Eq. (24), we group it into two equations as follows:

$$\frac{\partial^2 X}{\partial x^2} - \lambda X = 0, \quad (26)$$

$$\frac{\partial^2 Y}{\partial y^2} - \beta Y = 0, \quad (27)$$

where $\lambda$ and $\beta$ are constants and satisfy the relationship $\lambda + \beta + q = 0$. In this context, we investigate the long wavelength nonlinear waves whose amplitude is small but finite. By employing the reductive perturbation theory, we use the stretched coordinates of $\xi = \epsilon(x - ct)$, $\tau = \epsilon^2 t$, where $c$ is the linear phase speed. Then we expand the variables

$$N_1 = \epsilon^2 N_{11} + \epsilon^4 N_{12} + \cdots, \quad (28)$$

$$u = \epsilon^2 u_1 + \epsilon^4 u_2 + \cdots, \quad (29)$$

$$\psi = \epsilon^2 \psi_1 + \epsilon^4 \psi_2 + \cdots. \quad (30)$$

The parameter $\epsilon$ in the present study is in fact the order of the long wavelength number $k$ ($\epsilon \sim k$). We only consider the long wavelength wave, $k << 1$, or $\epsilon << 1$. However, the results of the reductive perturbation method may be appropriate even when the value of the parameter $\epsilon$ is as large as $\epsilon \sim 0.9$ for some system parameters [35].

The viscosity coefficient of a typical plasma is fairly small and even much less than $\epsilon$. For simplicity, we assume that $\nu = \epsilon \nu'$, where $\nu'$ is of the order of 1. The correctness of the reductive perturbation method has been verified previously in many branches of physics. The reductive perturbation theory indicates that all the physical quantities must be extended to the power of $\epsilon$; therefore, the viscosity coefficient of a plasma must be extended to the power of $\epsilon^4$. Here in the paper we choose $n = 3$.

Substituting Eqs. (20)-(22) into Eqs. (4)-(6) and solving the lowest order of the above expansions, we obtain the quantities as $n_1 = \frac{1}{\psi} \psi_1, N_{11} = \frac{1}{\psi} \psi_1$ and $c^2 = \frac{1}{1+\kappa^2}$. Proceeding to the next higher order of $\epsilon$, we obtain the KdV-type equation

$$\frac{\partial \psi_1}{\partial \tau} + A\psi_1 \frac{\partial \psi_1}{\partial \xi} + B\frac{\partial^3 \psi_1}{\partial \xi^3} + C\psi_1 = 0; \quad (31)$$

the coefficients $A$, $B$ and $C$, characterizing the nonlinear term, dispersive term and damping rate, are, correspondingly,

$$A = \frac{3}{2 \epsilon^3} X(x)Y(y), \quad (32)$$

$$B = \frac{\epsilon^3}{2}, \quad (33)$$

$$C = \frac{\nu'}{2} (\lambda + \beta). \quad (34)$$

For convenience, we further assume that $\psi_1 = \frac{6B^{1/3}}{\lambda} \psi_1', \tau = \tau'$ and $\xi = B^{1/3} \xi'$, then Eq. (31) can be reformulated as

$$\frac{\partial \psi_1'}{\partial \tau'} + 6\lambda \psi_1' \frac{\partial \psi_1'}{\partial \xi'} + \frac{\partial^3 \psi_1'}{\partial \xi'^3} + C\psi_1' = 0; \quad (35)$$

one of the approximate solutions of Eq. (35) is

$$\psi_1' = a(\tau') \text{sech}^2[(\frac{a_0}{2})^4 e^{(-2/3)D\tau'} \xi'], \quad (36)$$

$$a(\tau') = a_0 \exp(-\frac{4}{3} D\tau'). \quad (37)$$

The system of Eqs. (31)-(35) should be accompanied by boundary conditions, which will be specified in the next three sections. In particular, the first boundary condition stands for the situation where the electric potential at the boundaries is zero, or a constant, while the second boundary condition describes the case for which the electric field is zero. The third boundary condition is the other cases which are neither the first boundary nor the second boundary case.
3 The first boundary condition

It is found that the damping phenomena of the solitary waves are caused by many factors such as the boundary condition, the plasma viscosity, collisions, etc. The problem is thus reduced to that of solving $\lambda$ and $\beta$ according to Eq. (34). In this section, we introduce the first boundary condition as $\varphi = 0$ when $x = 0$, $x = a$ and $y = 0$, $y = b$; that is to say, the electric potential at the boundary is a constant. Hence, we have the boundary condition

$$X(0) = X(a) = Y(0) = Y(b) = 0;$$  \hspace{1cm} (38)

we assume that the solutions to Eqs. (26) and (27) can be written as

$$X(x) = D_1 \cos(\sqrt{-\lambda}x) + D_2 \sin(\sqrt{-\lambda}x),$$ \hspace{1cm} (39)

$$Y(y) = D'_1 \cos(\sqrt{-\beta}y) + D'_2 \sin(\sqrt{-\beta}y).$$ \hspace{1cm} (40)

Substituting Eq. (38) into Eq. (39) and Eq. (40), we have

$$\lambda = -\frac{n^2 \pi^2}{a^2},$$ \hspace{1cm} (41)

$$X(x) = \sum_{n=1}^{\infty} D_n \cos\left(\frac{n\pi x}{a}\right),$$ \hspace{1cm} (42)

$$\beta = -\frac{m^2 \pi^2}{b^2},$$ \hspace{1cm} (43)

$$Y(y) = \sum_{m=1}^{\infty} D'_m \sin\left(\frac{m\pi y}{b}\right).$$ \hspace{1cm} (44)

Substituting Eqs. (41)-(44) into Eqs. (32)-(34), the coefficients become

$$A = \frac{3}{2c} \sum_{n=1}^{\infty} D_n \cos\left(\frac{n\pi x}{a}\right) \sum_{m=1}^{\infty} D'_m \sin\left(\frac{m\pi y}{b}\right);$$ \hspace{1cm} (45)

$$B = \frac{\nu^3}{2};$$ \hspace{1cm} (46)

for $n = 1$, $m = 1$, we have in particular that

$$C = \frac{\nu'}{2} \left(\frac{\pi^2}{a^2} + \frac{\pi^2}{b^2}\right).$$ \hspace{1cm} (47)

The results indicate that $C$ is proportional to $\nu'$ and inversely proportional to $a^2$ and $b^2$.

4 The second boundary condition

In this section, we consider the problem of solitary wave propagation in a fully smooth boundary condition, which is the case where the electric field is zero.

For the case $x = 0$, $x = a$, we have

$$\varphi_x = 0,$$ \hspace{1cm} (48)

and for the case $y = 0$, $y = b$,

$$\varphi_y = 0,$$ \hspace{1cm} (49)

which implies that the boundary is smooth.

According to Eqs. (39)-(40), we have

$$X(x)' = -D_1 \sqrt{-\lambda} \sin(\sqrt{-\lambda}x) + D_2 \sqrt{-\lambda} \cos(\sqrt{-\lambda}x),$$ \hspace{1cm} (50)

$$Y(y)' = -D'_1 \sqrt{-\beta} \sin(\sqrt{-\beta}y) + D'_2 \sqrt{-\beta} \cos(\sqrt{-\beta}y).$$ \hspace{1cm} (51)

Substituting Eqs. (48)-(49) into Eqs. (50)-(51), we obtain

$$\lambda = -\frac{n^2 \pi^2}{a^2},$$ \hspace{1cm} (52)

$$X(x) = \sum_{n=1}^{\infty} D_n \cos\left(\frac{n\pi x}{a}\right),$$ \hspace{1cm} (53)

$$\beta = -\frac{m^2 \pi^2}{b^2},$$ \hspace{1cm} (54)

$$Y(y) = \sum_{m=1}^{\infty} D'_m \cos\left(\frac{m\pi y}{b}\right);$$ \hspace{1cm} (55)

therefore, Eqs. (32)-(34) become

$$A' = \frac{3}{2c} \sum_{n=1}^{\infty} D_n \cos\left(\frac{n\pi x}{a}\right) \sum_{m=1}^{\infty} D'_m \cos\left(\frac{m\pi y}{b}\right),$$ \hspace{1cm} (56)

$$B' = \frac{\nu^3}{2},$$ \hspace{1cm} (57)

and for $n = 1$ and $m = 1$

$$C' = \frac{\nu'}{2} \left(\frac{\pi^2}{a^2} + \frac{\pi^2}{b^2}\right).$$ \hspace{1cm} (58)

From Eq. (58), it is easy to find that this equation is the same as Eq. (47), indicating that the first and the second boundary conditions result in the same damping rate $C$.

5 The third boundary condition

Based on the above discussion, we now investigate the third boundary condition for the following two cases.

For the first case, $x = 0$ and $x = a$, we have

$$\varphi + L\varphi_x = 0.$$ \hspace{1cm} (59)

For the second case, $y = 0$ and $y = b$, we have

$$\varphi + L\varphi_y = 0,$$ \hspace{1cm} (60)

where $L$ is a constant; then, substituting Eqs. (59)-(60) into Eqs. (50)-(51), we have the relationship

$$\cot(\sqrt{-\lambda}a) = -\frac{1 + \lambda L^2}{2L\sqrt{-\lambda}},$$ \hspace{1cm} (61)

$$\cot(\sqrt{-\beta}b) = -\frac{1 + \beta L^2}{2L\sqrt{-\beta}},$$ \hspace{1cm} (62)

and assuming that $\eta = \cot \mu$, $\mu = \sqrt{-\lambda}a$, $\gamma = \cot \alpha$, $\alpha = \sqrt{-\beta}b$, we have $\eta = \frac{\mu}{4} (\frac{L\mu}{\alpha} - \frac{\alpha}{L\mu})$ and $\gamma = \frac{\gamma}{2} (\frac{\beta\gamma}{\beta} - \frac{\beta}{\gamma\beta})$.
or solitary wave. On the other hand, the larger the rect-

viscosity of the plasma, the stronger the damping of the

damping solitary wave. It is found that the larger the

\[ \lambda_n = -\frac{1}{a^2}\mu_n^2, \quad (63) \]

\[ X_n(x) = \sum_{n=1}^{\infty} D_n(\sin\sqrt{-\lambda_n}x + L\sqrt{-\lambda_n}\cos\sqrt{-\lambda_n}x), \]

\[ \beta_m = -\frac{1}{b^2}\nu_m, \quad (65) \]

\[ Y_m(y) = \sum_{m=1}^{\infty} D'_m(\sin\sqrt{-\nu_m}y + L\sqrt{-\nu_m}\cos\sqrt{-\nu_m}y). \]

Consequently, Eqs. (38)-(40) can be reformulated as

\[ A'' = \frac{3}{2c} \sum_{n=1}^{\infty} D_n(\sin\sqrt{-\lambda_n}x + L\sqrt{-\lambda_n}\cos\sqrt{-\lambda_n}x) \]

\[ \times \sum_{m=1}^{\infty} D'_m(\sin\sqrt{-\nu_m}y + L\sqrt{-\nu_m}\cos\sqrt{-\nu_m}y), \]

\[ B'' = \frac{c^3}{2}; \quad (68) \]

when \( n = 1 \) and \( m = 1 \), we obtain

\[ C'' = \frac{\nu'}{2}(\frac{\mu^2}{a^2} + \frac{\gamma_1^2}{b^2}), \quad (69) \]

which indicates that the damping rate \( C \) not only relies on parameters \( \nu', a \) and \( b \), but also depends on parameters \( \mu_1 \) and \( \gamma_1 \). In other words, the results indicate that the magnitude of the damping rate is dominated by the types of boundary condition.

In particular, when \( L = 0 \) or \( L \to \infty \), Eq. (69) turns out to be \( C'' = \frac{\nu'}{2}(\frac{\mu^2}{a^2} + \frac{\gamma_1^2}{b^2}) \). This is the same as Eqs. (47) and (58).

6 Discussion and conclusion

In summary, we have already obtained the KdV equation in the form of Eq. (35); specifically, if \( \nu' \to 0 \) or \( a \to \infty \) and \( b \to \infty \), the KdV equation becomes a standard KdV equation, which is given by

\[ \frac{\partial \psi'_1}{\partial t} + 6\psi'_1 \frac{\partial \psi'_1}{\partial x} + \frac{\partial^3 \psi'_1}{\partial x^3} = 0. \quad (70) \]

We have already investigated this case in the past literature [36-40].

In the general case of \( C \neq 0 \), Eq. (35) represents a damping solitary wave. It is found that the larger the viscosity of the plasma, the stronger the damping of the solitary wave. On the other hand, the larger the rectangle length \( a \) and width \( b \), the weaker the damping of the solitary wave. Besides, the intensities of the damping rate under the first and second boundary conditions are equal. Moreover, the solitary wave is damping as it propagates with time; in the limited case, the damping rate approaches infinity as the values of the length


and width of the cuboid approach zero or the viscosity coefficient of the plasma approaches infinity, which indicates that the magnitude of the solitary wave can be neglected.

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