Dispersion Relation of Linear Waves in Quantum Magnetoplasmas

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Abstract The quantum magnetohydrodynamic (QMHD) model is applied in investigating the propagation of linear waves in quantum magnetoplasmas. Using the QMHD model, the dispersion equation for quantum magnetoplasmas and the dispersion relations of linear waves are deduced. Results show that quantum effects affect the propagation of electron plasma waves and extraordinary waves (X waves). When we select the plasma parameters of the laser-based plasma compression (LBPC) schemes for calculation, the quantum correction cannot be neglected. Meanwhile, the corrections produced by the Fermi degeneracy pressure and Bohm potential are compared under different plasma parameter conditions.

Keywords: quantum plasmas, dispersion relation, QMHD, linear waves

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(Some figures may appear in colour only in the online journal)

1 Introduction

The linear wave in plasmas reflects the collective motions of plasmas, and it is the coupling of charged particles and self-consistent fields. Due to the long-range interaction of charged particles, as well as the different responses to magnetic fields by electrons and ions, the fluctuation models in plasmas are more complex than the ones in the neutral gas. Plasma waves exist and are widely applied in astrophysics and space physics, such as burst, radiation and aurora. The propagation and reflection of electromagnetic waves are the keys to guaranteeing and improving the quality of the radio communication. Plasma waves can also be used to diagnose the controlled thermonuclear fusion experiments without interference, such as the detection of the particle density, temperature and the thermal fluctuation of high temperature plasma. Therefore, the researches on the propagation of linear waves in plasmas are of high theoretical and practical significance.

Recently, quantum plasma has been a rapidly developing field and has been widely studied [1−5]. The quantum effects become significant in dense plasmas when the de Broglie wavelength of electrons becomes comparable to the spatial scale of the plasma system [6,7]. Under such circumstances, fermions will occupy different energy levels due to the Pauli Exclusion Principle. Fermi-Dirac substituted Boltzmann in the distribution function of electrons and the classical thermal pressure is replaced by the degeneracy pressure. Meanwhile, the electron tunneling effect represented by the Bohm potential [8,9] becomes significant when the spatial scale of the plasma system comes close to the interparticle distance of electrons. Researches indicate that quantum plasmas widely exist in ultrasmall semiconductor devices [10], astrophysics [11,12], and laser-plasma interaction [13].

The propagation of linear waves will be changed because the dynamics behavior of plasmas is affected by the quantum effects. Hence, we apply the QMHD model in investigating the propagation of linear waves in quantum magnetoplasmas. The dispersion relation of linear waves in quantum magnetoplasmas will be derived and compared with the ones in classical plasmas. Adopting different real plasma parameters, the quantum corrections will be calculated. In section 2, starting from the momentum equation, continuity equation and Maxwell’s equations, the dispersion equation for quantum magnetoplasmas is deduced. In section 3, the dispersion relations of linear waves are discussed with the case of propagating without an external magnetic field, as well as the cases of propagating parallel or perpendicular to the background magnetic field. In section 4, the contributions of the Bohm potential, and the Fermi statistics pressure are calculated and compared with different plasma parameters.

2 Dispersion equation for quantum magnetoplasmas

If we consider a zero-temperature quantum plasma, the dynamics of the electrons satisfy the momentum
\[
\left( \frac{\partial}{\partial t} + \mathbf{v} \cdot \nabla \right) \mathbf{v} = -\frac{1}{m} \left( \mathbf{E} + \frac{\mathbf{v} \times \mathbf{B}}{c} \right) - \nabla P \left( \frac{\hbar^2}{2m^2} \nabla^2 \sqrt{n} \right),
\]
and the continuity equation
\[
\frac{\partial n}{\partial t} + \nabla \cdot (nv) = 0,
\]
where \( n \) is the electron number density, and \( \mathbf{v} \) and \( m \) denote the fluid velocity and mass of electrons. The Fermi pressure gradient \( \nabla P \) is defined as \( \nabla P = m v_{te}^2 \nabla n/3 \), where \( v_{te} = \left( 3 \pi^2 n_0 \right)^{1/3} h/m \) is the Fermi velocity of electrons, and \( n_0 \) is the equal equilibrium number density of electrons.

Assuming every quantity \( \psi \) in Eq. (1) can be written as
\[
\psi = \psi_0 + \psi_1,
\]
where \( \psi_0 \) and \( \psi_1 (\psi_1 \ll \psi_0) \) are the equilibrium and perturbation values respectively. Here we assume that \( \mathbf{E}_0 = 0, \mathbf{v}_0 = 0, \mathbf{k} = (k, 0, 0), \) and \( \mathbf{B}_0 = (B_0 \cos \theta, 0, B_0 \sin \theta) \), where \( \theta \) is the angle between external magnetic field \( \mathbf{B}_0 \) and wave-vector \( \mathbf{k} \). With the above assumption, we obtain the linearized momentum equation
\[
\frac{\partial \mathbf{v}_1}{\partial t} = -\frac{1}{m} \left( \mathbf{E}_1 + \frac{\mathbf{v}_1}{c} \times \mathbf{B}_0 \right)
\]
where \( \mathbf{E}_1 \) and \( \mathbf{B}_1 \) satisfy the linearized Maxwell equations
\[
\begin{align*}
\nabla \times \mathbf{E}_1 &= -\frac{1}{c} \frac{\partial \mathbf{B}_1}{\partial t}, \\
\nabla \times \mathbf{B}_1 &= \frac{1}{c} \frac{\partial \mathbf{E}_1}{\partial t} + \frac{4\pi e_n}{c} \mathbf{v}_1.
\end{align*}
\]
Supposing the perturbation is proportional to \( \exp[i (\mathbf{k} \cdot \mathbf{r} - \omega t)] \), Eqs. (4) and (5) become
\[
\begin{align*}
-\mathbf{i} \omega \mathbf{v}_1 &= -\frac{e}{m} \left( \mathbf{E}_1 + \frac{\mathbf{v}_1}{c} \times \mathbf{B}_0 \right) - i \frac{\hbar^2 k_1^2 n_1}{3n_0} \mathbf{k} - i \frac{\hbar^2 k_1^2 n_1}{4m^2 n_0} \mathbf{k},
\end{align*}
\]
and
\[
\begin{align*}
n_1 &= \frac{\mathbf{k} \cdot \mathbf{v}_1}{\omega} n_0.
\end{align*}
\]
The three components of the fluid velocity \( \mathbf{v}_1 \) are calculated as
\[
\begin{align*}
v_x &= -\frac{ie}{\omega m (1-\Delta)} \left[ v_x + \frac{\omega_c \sin \theta \left( -i \omega E_y - \omega_c E_z \cos \theta + \frac{\hbar^2 E_z}{4m c^2} \sin \theta \right)}{\omega^2 - \omega_c^2 \cos^2 \theta - \frac{\hbar^2 E_z}{4m c^2} \sin^2 \theta} \right], \\
v_y &= \frac{e}{m} \left[ v_y - \frac{\omega E_y - \omega_c E_z \cos \theta + \frac{\hbar^2 E_z}{4m c^2} \sin \theta}{\omega^2 - \omega_c^2 \cos^2 \theta - \frac{\hbar^2 E_z}{4m c^2} \sin^2 \theta} \right], \\
v_z &= -\frac{ie}{\omega m} \left[ v_z - \frac{\omega_c \cos \theta \left( -i \omega E_y - \omega_c E_z \cos \theta + \frac{\hbar^2 E_z}{4m c^2} \sin \theta \right)}{\omega^2 - \omega_c^2 \cos^2 \theta - \frac{\hbar^2 E_z}{4m c^2} \sin^2 \theta} \right],
\end{align*}
\]
where \( \omega_c = eB_0/mc \) is the Larmor frequency of electrons, and \( \Delta = \frac{k^2 \omega^2}{\omega_c^2} + \frac{\hbar^4}{8mc \omega_c} \) is the quantum correction.

From the dispersion equations, we get the dispersion equation for plasmas as
\[
\det \left| \mathbf{k} \mathbf{k} - \mathbf{k}^2 \mathbf{I} + \frac{\omega^2}{c^2} \mathbf{e} \right| = 0.
\]
The current density \( \mathbf{j} \) is
\[
\mathbf{j} = -en_0 \mathbf{v}_1 = \sigma \cdot \mathbf{E}_1,
\]
and the dielectric tensor \( \varepsilon \) is
\[
\varepsilon = 1 + \frac{4\pi i}{\omega} \sigma.
\]
Solving Eqs. (10)-(13), the dispersion equation for quantum magnetoplasmas is obtained as
\[
\begin{align*}
\det \left| \begin{array}{ccc}
\frac{\omega^2}{c^2} - \frac{\omega_c^2}{c^2(1-\Delta)} & 1 + \frac{\omega_c^2 \sin^2 \theta}{\Omega^2 (1-\Delta)} & \frac{i \omega_c}{\sqrt{c^2(1-\Delta)}} \\
\frac{i \omega_c}{\sqrt{c^2(1-\Delta)}} & \frac{\omega_c^2 \sin \theta}{c^2} & \frac{\omega_c^2 \sin \theta \cos \theta}{\Omega^2} \\
\frac{-i \omega_c}{\sqrt{c^2(1-\Delta)}} & \frac{\omega_c^2 \sin \theta}{c^2} & \frac{\omega_c^2 \sin \theta}{c^2(1-\Delta)} - \frac{\omega_c^2 \sin \theta \cos \theta}{\Omega^2}
\end{array} \right| = 0,
\end{align*}
\]
where \( \Omega^2 = \omega_c^2 - \omega_c^2 \cos^2 \theta - \frac{\omega_c^2}{c^2} \sin^2 \theta, \omega_p = \left( 4\pi n_0 e^2/m \right)^{1/2} \) is the plasma frequency.
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3 Dispersion relation

In this section, the dispersion relations are discussed in the following cases: propagation without external magnetic field, propagation parallel to, and propagation perpendicular to the background magnetic field.

3.1 Without external magnetic field \((B_0 = 0)\)

When there is no external magnetic field, the dispersion equation Eq. (14) reduces to

\[
\begin{vmatrix}
\frac{\omega^2}{c^2} - \frac{\omega^2_p}{c^2(1-\Delta)} & 0 & 0 \\
0 & -k^2 + \frac{\omega^2}{c^2} \left(1 - \frac{\omega^2}{\omega^2_p}\right) & 0 \\
0 & 0 & -k^2 + \frac{\omega^2}{c^2} \left(1 - \frac{\omega^2}{\omega^2_p}\right)
\end{vmatrix} = 0. \tag{15}
\]

By solving Eq. (15), we obtain the dispersion relation of electron plasma waves as

\[
\omega^2 = \omega^2_p + \frac{1}{3} k^2 e^2 + \frac{\hbar^2 k^4}{4m^2}, \tag{16}
\]

when setting \(\hbar \to 0\), Eq. (16) reduces to the dispersion relation of Langmuir oscillations in the classical plasmas.

The second solution of Eq. (14) gives the dispersion relation of electromagnetic waves in quantum plasmas:

\[
\omega^2 = \omega^2_p + k^2 c^2. \tag{17}
\]

Eq. (17) indicates that the propagation of electromagnetic waves is not affected by quantum effects. The above results are consistent with the ones in our previous work, in which the dispersion relation and Landau damping of electron plasma waves and electromagnetic waves are derived by using the quantum kinetic model \cite{16}.

3.2 Parallel propagation \((\theta = 0^\circ)\)

When the propagation of waves is parallel to the external magnetic field, the dispersion equation Eq. (14) becomes

\[
\begin{vmatrix}
\frac{\omega^2}{c^2} - \frac{\omega^2_p}{c^2(1-\Delta)} & 0 & 0 \\
0 & -k^2 + \frac{\omega^2}{c^2} \left(1 - \frac{\omega^2}{\omega^2_p}\right) & 0 \\
0 & 0 & -\frac{\omega^2}{c^2} \left(1 - \frac{\omega^2}{\omega^2_p}\right)
\end{vmatrix} = 0. \tag{18}
\]

By solving Eq. (18), we obtain the dispersion relation of electron plasma waves:

\[
\omega^2 = \omega^2_p + \frac{1}{3} k^2 e^2 + \frac{\hbar^2 k^4}{4m^2} \tag{19}
\]

Comparing Eq. (19) with Eq. (16), we can make conclusion that the parallel external magnetic field does not affect the propagation of electron plasma waves.

The second solution of Eq. (18) is

\[
\frac{c^2 k^2}{\omega^2} = 1 - \frac{\omega^2_p}{\omega^2(\omega \pm \omega_c)}, \tag{20}
\]

which is the dispersion relation of left-handed wave (L waves) and right-handed wave (R waves) in quantum magnetoplasmas. It also indicates that the propagations of L and R waves are not affected by quantum effects.

3.3 Perpendicular propagation \((\theta = 90^\circ)\)

When the propagation of waves is perpendicular to the external magnetic field, the dispersion equation Eq. (35) becomes

\[
\begin{vmatrix}
\frac{\omega^2}{c^2} \left[1 - \frac{\omega^2}{\omega^2(1-\Delta)}\right] & \frac{\omega^2_p}{c^2(1-\Delta)} & 0 \\
-\frac{\omega^2_p}{c^2} & -k^2 + \frac{\omega^2}{c^2} \left(1 - \frac{\omega^2}{\omega^2_p}\right) & 0 \\
0 & 0 & -k^2 + \frac{\omega^2}{c^2} \left(1 - \frac{\omega^2}{\omega^2_p}\right)
\end{vmatrix} = 0. \tag{21}
\]

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The first solution of Eq. (21) is
\[ \omega^2 = k^2 c^2 + \omega_p^2, \]
which is the dispersion relation of ordinary waves (O wave). Comparing Eq. (22) with Eq. (17), it is found that the external magnetic field perpendicular to the wave-vector does not affect the propagation of electromagnetic waves. It also indicates that the propagation of O waves is not affected by quantum effects.

The second solution of Eq. (21) is
\[ \frac{c^2 k^2}{\omega^2} = 1 - \frac{\omega^2}{\omega_p^2} (1 - \Delta) - \frac{\omega^2}{\omega_h^2} (1 - \Delta) - \frac{\omega^2}{c^2}, \]
which is the dispersion relation of extraordinary waves (X wave) in quantum magnetoplasmas, and
\[ \omega_h^2 = \omega_p^2 + \omega_c^2, \]
is the upper hybrid oscillation.

### 4 Calculation and discussion

In quantum plasmas, quantum effects become significant when the de Broglie wavelength \( \lambda_B \) of electrons becomes comparable to the spatial scale \( d \) of the plasma system, where \( \lambda_B = h/mv_{th} \) and \( d \sim n_0^{-1/3} \). According to the quantum condition of \( \lambda_B n_0 \geq 1 \), we have
\[ \frac{n_0}{T^{3/2}} \geq \left( \frac{\sqrt{mk_B}}{h} \right)^3 \sim 10^{16} \text{ cm}^{-3}/\text{K}^{3/2}. \] (24)

In this section, we make quantitative calculations of quantum effects by adopting the parameters of the laser-based plasma compression (LBPC) schemes. The plasmas parameters of LBPC are \( n_0 \sim 10^{27} \text{ cm}^{-3} \) and \( T \sim 10 \text{ keV} \). Obviously, the above parameters satisfy the quantum condition of \( \lambda_B^3 n_0 \geq 1 \).

**Fig. 1** The dispersion curves of Langmuir waves in classical and quantum plasmas are represented by solid and dashed lines respectively. The plasma parameter is \( n_0 \sim 10^{27} \text{ cm}^{-3} \).

The quantum correction to the dispersion relation of plasma waves is shown in Fig. 1. It indicates that the Langmuir oscillations can propagate in cold plasmas due to quantum effects, and the quantum corrected term can reach \( 10^{-1} \) when \( k \approx 3 \times 10^8 \text{ cm}^{-1} \).

The quantum corrections produced by Fermi pressure and Bohm potential are compared in Fig. 2 and Fig. 3, which show that in the long wavelength and low density regime, the correction produced by Fermi pressure is greater than the Bohm potential. In the short wavelength and high density regime, the particle-like nature of waves becomes obvious and the electron tunneling effect (Bohm potential) becomes more significant than Fermi pressure.

**Fig. 2** The curve represents the ratio of quantum corrections produced by Fermi pressure and Bohm potential, where \( k = 3 \times 10^9 \text{ cm}^{-1} \), \( c_F \) and \( c_B \) denote the corrections produced by Fermi pressure and Bohm potential respectively.

**Fig. 3** The curve shows the ratio of quantum corrections produced by Fermi pressure and Bohm potential, where \( n_0 = 10^{27} \text{ cm}^{-3} \), \( c_F \) and \( c_B \) denote the corrections produced by Fermi pressure and Bohm potential respectively.

**Fig. 4** presents the dispersion curves of X waves in classical and quantum plasmas respectively. It shows that the quantum correction is significant when we chose the plasma parameters \( n_0 = 10^{27} \text{ cm}^{-3} \) and \( B_0 = 10^9 \text{ Gs} \).
Fig. 4 The dependences of dispersion relation of X waves on the wave-number $k$. The cases of classical and quantum are represented by the solid line and dashed line, respectively. The plasma parameters are $n_0 = 10^{27} \text{ cm}^{-3}$ and $B_0 = 10^9 \text{ Gs}$.

Due to the extensive application in astrophysics, laser-plasma interaction and inertial confinement fusion, quantum plasmas has received wide attention. In this paper, we present a theoretical investigation on the propagation of linear waves in the quantum magnetoplasmas. Using the QMHD model, the dispersion equation for quantum magnetoplasmas and the dispersion relations of linear waves are deduced. When we select the plasma parameters of the laser-based plasma compression (LBPC) schemes for calculation, the quantum correction cannot be neglected. The investigation on the propagation of linear waves in quantum plasmas is the consummation to the present theoretical researches of quantum plasmas, and it is significant to laser-plasma interaction experiments and inertial confinement fusion.

References


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